

# Meson vacuum phenomenology in a three-flavor linear sigma model with (axial-)vector mesons

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We study scalar, pseudoscalar, vector, and axial-vector mesons with non-strange and strange quantum numbers in the framework of a linear sigma model with global chiral  $U(N_f)_L \times U(N_f)_R$  symmetry. We perform a global fit of meson masses, decay widths, as well as decay amplitudes. The quality of the fit is, for a hadronic model that does not consider isospin-breaking effects, surprisingly good. We also investigate the question whether the scalar  $\bar{q}q$  states lie below or above 1 GeV and find the scalar states above 1 GeV to be preferred as  $\bar{q}q$  states. Additionally, we also describe the axial-vector resonances as  $\bar{q}q$  states.

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## I. INTRODUCTION

Understanding the meson mass spectrum in the region below 2 GeV is one of the fundamental problems of QCD. While the quark model seems to work very well for many resonances [see, for instance, the summary in Ref. [1]], some fundamental questions, such as the constituent quark content of scalar and axial-vector resonances, are still unanswered.

The quark content of the scalar mesons has been a matter of debate for many decades [2]. The Particle Data Group (PDG) [1] suggests the existence of five  $I(J^{PC}) = 0(0^{++})$  states in the region below 1.8 GeV:  $f_0(500)$ ,  $f_0(980)$ ,  $f_0(1370)$ ,  $f_0(1500)$ , and  $f_0(1710)$ . Also the existence of a sixth state,  $f_0(1790)$ , close to, but distinct from,  $f_0(1710)$  has been claimed [3]. Additionally, there are scalar resonances in the isotriplet and in the isodoublet sector as well: the  $I = 1$  states  $a_0(980)$  and  $a_0(1450)$  are well established; the existence of the  $I = 1/2$  state  $K_0^*(800)$  (or  $\kappa$ ) is confirmed by some [4] and disputed by other authors [5], whereas the  $I = 1/2$  scalar kaon,  $K_0^*(1430)$ , is an established resonance.

A description of all mentioned scalar states as  $\bar{q}q$  states is not possible for a simple reason: the number of physical resonances is much larger than the number of resonances that can be constructed within a  $\bar{q}q$  picture of mesons. Explicitly, there is only one  $I(J^{PC}) = 0(0^{++})$  state that can be constructed from the (non-strange)  $u$  and  $d$  quarks (provided they are degenerate). Denoting this state as  $\sigma_N$  we obtain  $\sigma_N \equiv (\bar{u}u + \bar{d}d)/\sqrt{2}$ . An additional  $I(J^{PC}) = 0(0^{++})$  state can be constructed, if the strange quark is considered as well: the pure strange state  $\sigma_S \equiv \bar{s}s$ . Since  $\sigma_N$  and  $\sigma_S$  have the same  $I(J^{PC})$  quantum numbers, we expect the physical spectrum to consist of mixed, rather than pure, states of  $\sigma_N$  and  $\sigma_S$ ; however, such mixing will, of course, produce exactly two new states. These states will correspond to at most two of the mentioned five (six)  $f_0$  states, and the natural question is then: *which two?*

Similarly, in both the  $I = 1$  and  $I = 1/2$  sectors one can construct only one quark-antiquark resonance. Restricting for example to electric charge +1, one has the states  $a_0^+ = \bar{d}u$  and  $K_0^{*+} = \bar{s}u$ . The state  $a_0^+ = \bar{d}u$  can be assigned to  $a_0(980)$  or  $a_0(1450)$ , and  $K_0^{*+} = \bar{s}u$  to  $K_0^*(800)$  or  $K_0^*(1430)$ . The question is: *which is the correct assignment?*

An answer to these questions is inevitably complicated for several reasons. Firstly, as already indicated, states

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with the same  $I(J^{PC})$  quantum numbers are expected to mix – this needs to be considered in particular in the scalar sector due to the large number of physical resonances. Secondly, basic features (pole mass and decay width) of some scalar mesons [e.g.  $f_0(500)$ ] are notoriously difficult to determine experimentally, which makes it non-trivial to determine the structure of these states [see, e.g. Ref. [6] for an example regarding the  $f_0(500)$  resonance and Ref. [7] for an example regarding  $f_0(1370)$ ].

The question of how to correctly determine the quark structure of the mesons is not only restricted to the scalar sector. More recently, also the nature of the axial-vector mesons, most notably  $a_1(1260)$ , but also that of the isoscalar states  $f_1(1285)$ ,  $f_1(1525)$ , and the kaonic state  $K_1(1270)$  [or  $K_1(1400)$ , see the discussion in Ref. [8] and refs. therein] have been investigated. Should one interpret the isotriplet resonance  $a_1(1260)$  as a quark-antiquark state, as the quark model suggests (e.g.  $a_1^+ = \bar{d}u$ ), or is this state a broad  $\rho\pi$  molecular state?

Understanding these issues is not only crucial for hadron vacuum spectroscopy but is also important at nonzero temperatures and densities, because the correct identification of the chiral partner of the pion and of the  $\rho$  meson is necessary for a proper description of the in-medium properties of hadrons [9]. In fact, if the  $a_1(1260)$  is (predominantly) a quark-antiquark state, it is the chiral partner of the  $\rho$  meson, with which it becomes degenerate at large  $T$  and  $\mu$  [10]. The chiral partner of the pion is the scalar-isoscalar state  $\sigma_N \equiv (\bar{u}u + \bar{d}d)/\sqrt{2}$  which has been identified with the lightest scalar state  $f_0(500)$  in many works. However, as various theoretical results for the vacuum have shown, and as we shall present in detail in this work, this assignment is not correct: it turns out that  $f_0(1370)$  emerges as the chiral partner of the pion.

Last but not least, the question of the origin of hadron masses is important. One aims to understand to which extent the quark condensate,  $\langle \bar{q}q \rangle$ , and the gluon condensate,  $\langle G^{\mu\nu}G_{\mu\nu} \rangle$ , generate the hadron masses. To give an example, which condensate is predominantly responsible for the mass of the  $\rho$  meson? And related to this, how will the mass of the  $\rho$  (and that of other resonances) change in the medium?

The answers to all these fundamental questions are in principle contained in the QCD Lagrangian. Unfortunately, QCD cannot be solved by analytic means from first principles in the low-energy domain. For this reason, effective theories have been developed which share some of the underlying symmetries of QCD. The QCD Lagrangian exhibits, in addition to the local  $SU(3)_c$  color symmetry and the discrete  $CPT$  symmetry, a global chiral  $U(N_f)_L \times U(N_f)_R \equiv U(1)_V \times U(1)_A \times SU(N_f)_V \times SU(N_f)_A$  symmetry which is broken in several ways: spontaneously [due to the chiral condensate  $\langle \bar{q}q \rangle = \langle \bar{q}_R q_L + \bar{q}_L q_R \rangle \neq 0$  [11]], explicitly (due to non-vanishing quark masses), as well as at the quantum level [the  $U(1)_A$  anomaly [12]].

In the framework of effective theories the chiral symmetry of QCD can be realized along two lines: linearly [13] and non-linearly [14]. In the former case one obtains the so-called linear sigma model: both scalar and pseudoscalar degrees of freedom are present and vectors as well as axial-vectors can be included into the model in a straightforward manner [15–18]. Note that an important advantage of linear sigma models is the possibility to investigate the state of matter at large values of temperature and chemical potential [e.g. in the region of the chiral transition, i.e., where chiral symmetry is restored [19, 20]]. Every linear sigma model contains so-called chiral partners – states that mix with each other under axial transformations – that become degenerate when chiral symmetry is restored. In the non-linear realization (i.e., the non-linear sigma models), the scalar and the (axial-)vector states are integrated out and the pseudoscalar states are the only degrees of the freedom: one obtains the Lagrangian of chiral perturbation theory [21]. If the vector mesons are not integrated out, one is left with chiral perturbation theory with vector mesons, see e.g. Ref. [22] and refs. therein.

In this paper, we present a linear sigma model containing scalar, pseudoscalar, vector, and axial-vector mesons with both non-strange and strange quantum numbers. Although this project represents a straightforward implementation of the principles of the linear realization of chiral symmetry as already outlined in Ref. [17], this is – to our knowledge – the first time that all these degrees of freedom are considered within a single linear chiral framework. In view of the large number of the fields involved, our model shall be referred to as “extended Linear Sigma Model”, or “eLSM”.

In Refs. [15, 23], we have already presented a linear sigma model with vector and axial-vector mesons for two flavors. Comparing our model with experimental data for meson vacuum phenomenology led us to conclude that the scalar  $\bar{q}q$  states are located in the energy region above 1 GeV in the hadron mass spectrum. Explicitly, we concluded that the resonances  $f_0(1370)$  and  $a_0(1450)$  are strongly favoured to be scalar quarkonia. The present work is more general: we now consider three flavors both in the (pseudo)scalar and (axial-)vector channels in order to ascertain whether the conclusion of Ref. [15] is still valid once strange mesons are considered. We emphasize that, as discussed in Ref. [15], all fields entering our model describe pure quark-antiquark states. The reason is that masses and decay widths of our

theoretical states scale as  $N_c^0$  and  $N_c^{-1}$ , respectively, where  $N_c$  is the number of colors; thus all decay widths vanish in the large- $N_c$  limit. Note that the inclusion of strange mesons provides us with a large number of very precise data [in general, decisively more precise than in the case of non-strange mesons [1]]. As a consequence, our model parameters are much more constrained than in Ref. [15]. Preliminary results of this work have already been presented in Ref. [24].

In order to (i) test the performance of a linear sigma model in describing the overall phenomenology of (pseudo)scalar and (axial-)vector mesons and (ii) ascertain which scalar states are predominantly quarkonia, we perform a global fit in which 21 experimentally known quantities (decay constants, masses, and decay widths as well as amplitudes) are included. The situation in the scalar-isoscalar sector is extremely uncertain because five (six) resonances,  $f_0(500)$ ,  $f_0(980)$ ,  $f_0(1370)$ ,  $f_0(1500)$ , and  $f_0(1710)$  [and, possibly in the future,  $f_0(1790)$ ] are listed in the PDG. In addition, the  $f_0(500)$  and  $f_0(1370)$  decay widths and branching ratios are poorly known and  $f_0(980)$  suffers from a distortion by the  $K\bar{K}$  threshold. Moreover, in the present version of the model the scalar glueball is not included, but it is known that the latter state with a bare mass of about 1.5 – 1.7 GeV as predicted by lattice QCD [25] can sizably affect the masses and the branching ratios of the scalar states above 1 GeV, that is  $f_0(1370)$ ,  $f_0(1500)$ , and  $f_0(1710)$  [26]. In view of all these reasons, we do *not* include the scalar-isoscalar states into the fit, but we test *a posteriori* their assignment and phenomenology. We do, however, include the isotriplet and isodoublet quark-antiquark scalar states and we test all four combinations  $a_0(1450) / K_0^*(1430)$ ,  $a_0(980) / K_0^*(800)$ ,  $a_0(980) / K_0^*(1430)$ , and  $a_0(1450) / K_0^*(800)$ . Quite remarkably, the outcome of the fit is univocal: only the pair  $a_0(1450) / K_0^*(1430)$  yields a good fit, while the other combinations do not. We thus conclude that the  $I = 1$  and  $I = 1/2$  quark-antiquark scalar resonances lie above 1 GeV. In fact, the quality of our fit is surprisingly good. We describe *all* experimental quantities with an average error of 5%, and most of them even to much better precision. We perceive this to be a remarkable achievement within an (in principle, quite simple) effective model for the strong interaction.

We then study the two isoscalar mesons of the model: a consequence of our fit is their mass in the large- $N_c$  limit. These masses turn out to be about 1.36 GeV and 1.53 GeV, respectively. Varying large- $N_c$  suppressed parameters which cannot be determined by our fit, one can also study their decays: the non-strange quark-antiquark state is well described by  $f_0(1370)$  and the heavier  $\bar{s}s$  state might be  $f_0(1710)$ . Turning to the axial-vector channel, both masses and decays are well described by assuming the axial-vector resonances  $a_1(1260)$ ,  $f_1(1285)$ ,  $f_1(1525)$ , and  $K_1(1270)$  as (predominantly) quark-antiquark states. Detailed calculations of all formulas used in this paper as well as further discussion are presented in Ref. [27].

The outline of the paper is as follows. In Sec. II we discuss the  $U(3)_L \times U(3)_R$  linear sigma model with vector and axial-vector mesons. In Sec. III we discuss the global fit and its consequences and in Sec. IV we present our conclusions. We defer detailed calculations to Appendices A and B. Our units are  $\hbar = c = 1$ ; the metric tensor is  $g_{\mu\nu} = \text{diag}(+, -, -, -)$ .

## II. THE MODEL

### A. Lagrangian

In this section we discuss our model: a linear sigma model with (axial-)vector mesons based on the chiral  $U(3)_L \times U(3)_R$  symmetry. We retain only operators up to fourth order<sup>1</sup>; higher-order operators cannot appear when dilatation symmetry is imposed, see the discussion in Refs. [28, 29]. The Lagrangian of the model reads

$$\begin{aligned}
\mathcal{L} = & \text{Tr}[(D_\mu \Phi)^\dagger (D_\mu \Phi)] - m_0^2 \text{Tr}(\Phi^\dagger \Phi) - \lambda_1 [\text{Tr}(\Phi^\dagger \Phi)]^2 - \lambda_2 \text{Tr}(\Phi^\dagger \Phi)^2 \\
& - \frac{1}{4} \text{Tr}(L_{\mu\nu}^2 + R_{\mu\nu}^2) + \text{Tr} \left[ \left( \frac{m_1^2}{2} + \Delta \right) (L_\mu^2 + R_\mu^2) \right] + \text{Tr}[H(\Phi + \Phi^\dagger)] \\
& + c_1 (\det \Phi - \det \Phi^\dagger)^2 + i \frac{g_2}{2} (\text{Tr}\{L_{\mu\nu}[L^\mu, L^\nu]\} + \text{Tr}\{R_{\mu\nu}[R^\mu, R^\nu]\}) \\
& + \frac{h_1}{2} \text{Tr}(\Phi^\dagger \Phi) \text{Tr}(L_\mu^2 + R_\mu^2) + h_2 \text{Tr}[(L_\mu \Phi)^2 + (\Phi R_\mu)^2] + 2h_3 \text{Tr}(L_\mu \Phi R^\mu \Phi^\dagger) \\
& + g_3 [\text{Tr}(L_\mu L_\nu L^\mu L^\nu) + \text{Tr}(R_\mu R_\nu R^\mu R^\nu)] + g_4 [\text{Tr}(L_\mu L^\mu L_\nu L^\nu) + \text{Tr}(R_\mu R^\mu R_\nu R^\nu)] \\
& + g_5 \text{Tr}(L_\mu L^\mu) \text{Tr}(R_\nu R^\nu) + g_6 [\text{Tr}(L_\mu L^\mu) \text{Tr}(L_\nu L^\nu) + \text{Tr}(R_\mu R^\mu) \text{Tr}(R_\nu R^\nu)] , \tag{1}
\end{aligned}$$

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<sup>1</sup> With the exception of the anomaly term which is of sixth order.

where

$$\begin{aligned} D^\mu \Phi &\equiv \partial^\mu \Phi - ig_1(L^\mu \Phi - \Phi R^\mu) - ieA^\mu[T_3, \Phi], \\ L^{\mu\nu} &\equiv \partial^\mu L^\nu - ieA^\mu[T_3, L^\nu] - \{\partial^\nu L^\mu - ieA^\nu[T_3, L^\mu]\}, \\ R^{\mu\nu} &\equiv \partial^\mu R^\nu - ieA^\mu[T_3, R^\nu] - \{\partial^\nu R^\mu - ieA^\nu[T_3, R^\mu]\}, \end{aligned}$$

The quantities  $\Phi$ ,  $R^\mu$ , and  $L^\mu$  represent the scalar and vector nonets:

$$\Phi = \sum_{i=0}^8 (S_i + iP_i)T_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{(\sigma_N + a_0^0) + i(\eta_N + \pi^0)}{\sqrt{2}} & a_0^+ + i\pi^+ & K_0^{*+} + iK^+ \\ a_0^- + i\pi^- & \frac{(\sigma_N - a_0^0) + i(\eta_N - \pi^0)}{\sqrt{2}} & K_0^{*-} + iK^0 \\ K_0^{*-} + iK^- & \bar{K}_0^{*0} + i\bar{K}^0 & \sigma_S + i\eta_S \end{pmatrix}, \quad (2)$$

$$L^\mu = \sum_{i=0}^8 (V_i^\mu + A_i^\mu)T_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega_N + \rho^0}{\sqrt{2}} + \frac{f_{1N} + a_1^0}{\sqrt{2}} & \rho^+ + a_1^+ & K^{*+} + K_1^+ \\ \rho^- + a_1^- & \frac{\omega_N - \rho^0}{\sqrt{2}} + \frac{f_{1N} - a_1^0}{\sqrt{2}} & K^{*0} + K_1^0 \\ K^{*-} + K_1^- & \bar{K}^{*0} + \bar{K}_1^0 & \omega_S + f_{1S} \end{pmatrix}^\mu, \quad (3)$$

$$R^\mu = \sum_{i=0}^8 (V_i^\mu - A_i^\mu)T_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega_N + \rho^0}{\sqrt{2}} - \frac{f_{1N} + a_1^0}{\sqrt{2}} & \rho^+ - a_1^+ & K^{*-} - K_1^- \\ \rho^- - a_1^- & \frac{\omega_N - \rho^0}{\sqrt{2}} - \frac{f_{1N} - a_1^0}{\sqrt{2}} & K^{*0} - K_1^0 \\ K^{*-} - K_1^- & \bar{K}^{*0} - \bar{K}_1^0 & \omega_S - f_{1S} \end{pmatrix}^\mu, \quad (4)$$

where the assignment to physical particles is shown as well<sup>2</sup>. Here,  $T_i$  ( $i = 0, \dots, 8$ ) denote the generators of  $U(3)$ , while  $S_i$  represents the scalar,  $P_i$  the pseudoscalar,  $V_i^\mu$  the vector, and  $A_i^\mu$  the axial-vector meson fields, and  $A^\mu$  is the electromagnetic field. It should be noted that here and below we use the so-called non strange – strange basis in the  $(0-8)$  sector, defined as

$$\begin{aligned} \varphi_N &= \frac{1}{\sqrt{3}} (\sqrt{2} \varphi_0 + \varphi_8), \\ \varphi_S &= \frac{1}{\sqrt{3}} (\varphi_0 - \sqrt{2} \varphi_8), \quad \varphi \in (S_i, P_i, V_i^\mu, A_i^\mu), \end{aligned} \quad (5)$$

which is more suitable for our calculations. Moreover,  $H$  and  $\Delta$  are constant external fields defined as

$$H = H_0 T_0 + H_8 T_8 = \begin{pmatrix} \frac{h_{0N}}{2} & 0 & 0 \\ 0 & \frac{h_{0N}}{2} & 0 \\ 0 & 0 & \frac{h_{0S}}{\sqrt{2}} \end{pmatrix}, \quad (6)$$

$$\Delta = \Delta_0 T_0 + \Delta_8 T_8 = \begin{pmatrix} \frac{\tilde{\delta}_N}{2} & 0 & 0 \\ 0 & \frac{\tilde{\delta}_N}{2} & 0 \\ 0 & 0 & \frac{\tilde{\delta}_S}{\sqrt{2}} \end{pmatrix} \equiv \begin{pmatrix} \delta_N & 0 & 0 \\ 0 & \delta_N & 0 \\ 0 & 0 & \delta_S \end{pmatrix}. \quad (7)$$

Throughout this work we assume exact isospin symmetry for  $u$  and  $d$  quarks, such that the first two diagonal elements in Eq. (6) and Eq. (7) are identical.

Let us now discuss the assignment of fields in Eqs. (2) – (4). If we consider isospin multiplets as single degrees of freedom, then there are 16 resonances that can be described by the model:  $\sigma_N$ ,  $\sigma_S$ ,  $\bar{a}_0$ ,  $K_0^*$  (scalar);  $\eta_N$ ,  $\eta_S$ ,  $\bar{\pi}$ ,  $K$  (pseudoscalar);  $\omega_N^\mu$ ,  $\omega_S^\mu$ ,  $\bar{\rho}^\mu$ ,  $K^{*\mu}$  (vector), and  $f_{1N}^\mu$ ,  $f_{1S}^\mu$ ,  $\bar{a}_1^\mu$ ,  $K_1$  (axial-vector). All fields in our model represent  $\bar{q}q$  states, as discussed in Ref. [15]. If we assign a state from our model to a physical resonance we, therefore, implicitly assume that this resonance is a  $\bar{q}q$  state. This assumption can be tested for a multitude of physical resonances in the scalar and axial-vector sectors, as discussed below (in the pseudoscalar and the vector channels, there are no ambiguities).

In the non-strange sector, we assign the fields  $\bar{\pi}$  and  $\eta_N$  to the pion and the non-strange part of the  $\eta$  and  $\eta'$  mesons,  $\eta_N \equiv (\bar{u}u + \bar{d}d)/\sqrt{2}$ . The fields  $\omega_N^\mu$  and  $\bar{\rho}^\mu$  represent the  $\omega(782)$  and  $\rho(770)$  vector mesons, respectively, and the fields

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<sup>2</sup> With the exception of the  $(0-8)$  sector where particle mixing takes place (see below).

$f_{1N}^\mu$  and  $\vec{a}_1^\mu$  represent the  $f_1(1285)$  and  $a_1(1260)$  mesons, respectively. In the strange sector, we assign the  $K$  fields to the kaons; the  $\eta_S$  field is the strange contribution to the physical  $\eta$  and  $\eta'$  fields [ $\eta_S \equiv \bar{s}s$ ]; the  $\omega_S$ ,  $f_{1S}$ ,  $K^*$ , and  $K_1$  fields correspond to the  $\phi(1020)$ ,  $f_1(1420)$ ,  $K^*(892)$ , and  $K_1(1270)$  [or  $K_1(1400)$ ] mesons, respectively.

Unfortunately, the assignment of the scalar fields is substantially less clear. Experimental data suggest existence of five (six) scalar-isoscalar states below 1.8 GeV:  $f_0(500)$ ,  $f_0(980)$ ,  $f_0(1370)$ ,  $f_0(1500)$ , and  $f_0(1710)$  [as well as  $f_0(1790)$ ]. Note that the existence of these five states is acknowledged by the Particle Data Group – PDG [1]. [The sixth state,  $f_0(1790)$ , will not be of importance for the rest of our work because its predominant  $\pi\pi$  decay renders it a putative radial excitation of  $f_0(1370)$ , and our model describes ground-state quarkonia only.]

Our model contains the pure non-strange isoscalar  $\sigma_N$  and the pure strange isoscalar  $\sigma_S$ . We will demonstrate below that our model yields mixing of  $\sigma_N$  and  $\sigma_S$ , producing a predominantly non-strange state labeled as  $f_0^L$ , and a predominantly strange state labeled as  $f_0^H$ . Assignment of the mentioned mixed states to physical resonances is ambiguous because, as already discussed, there are five physical states all of which could, in principle, be candidates for  $f_0^L$  and  $f_0^H$ .

Similarly, the isospin triplet  $a_0$  can be assigned to different physical resonances – although, in this case, there are only two candidate states:  $a_0(980)$  and  $a_0(1450)$ . An analogous statement holds for the scalar kaon  $K_0^*$  that can be assigned to the resonances  $K_0^*(800)$  or  $K_0^*(1430)$ . A preliminary examination of the assignment of the  $a_0$  field has been performed in Ref. [15] where it was concluded that it most likely corresponds to the  $a_0(1450)$  resonance [or, equivalently,  $a_0(1450)$  rather than  $a_0(980)$  was favored to represent a  $\bar{q}q$  state]. The discussion in Ref. [15] was limited to non-strange mesons. In this work, besides the assignment of  $a_0$ , we also have to consider possible assignments for the strange scalar field  $K_0^*$ ; there are also two possibilities: either  $K_0^*(800)$  or  $K_0^*(1430)$ .

In the Lagrangian (1) there are two terms which break the original  $U(3)_L \times U(3)_R [= U(3)_V \times U(3)_A]$  symmetry, namely the determinant term, which breaks the  $U(1)_A$  symmetry, and the explicit symmetry breaking terms of Eqs. (6) and (7), which break  $U(3)_A$ , if  $H_0, \Delta_0 \neq 0$  and  $U(3)_V \rightarrow SU(2)_V \times U(1)_V$ , if  $H_8, \Delta_8 \neq 0$  [for more details see, e.g. Ref. [19]]. Besides this explicit symmetry breaking the chiral symmetry is also broken spontaneously. If isospin symmetry is exact, only the  $\sigma_N$  and  $\sigma_S$  scalar fields, carrying the same quantum numbers as the vacuum, can have nonzero vacuum expectation values (vev's)<sup>3</sup>. Moreover, the parameter  $c_1$  describes the axial anomaly, i.e., the  $U(1)_A$  symmetry is explicitly broken by this term.

As a last remark about the Lagrangian we stress that, when including a dilaton field, only two terms are modified [in addition, one has to add the dilatation potential, for details see Ref. [28]]: the two parameters  $m_0^2$  and  $m_1^2$  should be replaced by

$$m_0^2 \rightarrow aG^2, \quad m_1^2 \rightarrow bG^2, \quad (8)$$

where  $G$  is the dilaton/glueball field and  $a$  and  $b$  are dimensionless constants. Obviously, upon condensation of the glueball field, one obtains  $m_0^2 = aG_0^2$  and  $m_1^2 = bG_0^2$ .

## B. Tree-level masses and mixing terms

After spontaneous symmetry breaking, the fields with nonzero vev's are shifted by their expectation values, namely,  $\sigma_{N/S} \rightarrow \sigma_{N/S} + \phi_{N/S}$ , where we have introduced  $\phi_{N/S} \equiv \langle \sigma_{N/S} \rangle$ . After substituting the shifted fields into the Lagrangian (1), one obtains the tree-level masses by selecting all terms quadratic in the fields,

$$\begin{aligned} \mathcal{L}^{\text{quad}} = & -\frac{1}{2} S_i [\delta_{ij} \square + (m_S^2)_{ij}] S_j \\ & -\frac{1}{2} P_i [\delta_{ij} \square + (m_P^2)_{ij}] P_j \\ & -\frac{1}{2} V_{i\mu} [(-g^{\mu\nu} \square + \partial^\mu \partial^\nu) \delta_{ij} - g^{\mu\nu} (m_V^2)_{ij}] V_{j\nu} \\ & -\frac{1}{2} A_{i\mu} [(-g^{\mu\nu} \square + \partial^\mu \partial^\nu) \delta_{ij} - g^{\mu\nu} (m_A^2)_{ij}] A_{j\nu} \end{aligned}$$

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<sup>3</sup> In case of isospin breaking, also  $\sigma_3$  would have a nonzero vev.

$$\begin{aligned}
& -\frac{1}{2}V_{i\mu}(ig_1f_{ijk}\phi_k\partial^\mu)S_j - \frac{1}{2}S_i(ig_1f_{ijk}\phi_k\partial^\nu)V_{j\nu} \\
& + \frac{1}{2}A_{i\mu}(g_1d_{ijk}\phi_k\partial^\mu)P_j - \frac{1}{2}P_i(g_1d_{ijk}\phi_k\partial^\nu)A_{j\nu},
\end{aligned} \tag{9}$$

where  $(m_S^2)_{ij}$ ,  $(m_P^2)_{ij}$ ,  $(m_V^2)_{ij}$ , and  $(m_A^2)_{ij}$  are the scalar, pseudoscalar, vector, and axial-vector (squared) mass matrices, respectively, see their explicit expressions in Appendix A. Moreover,  $f_{ijk}$  and  $d_{ijk}$  are the antisymmetric and symmetric structure constants of  $U(3)$ . The (squared) mass matrices are in general non-diagonal due to the mixing among particles sitting in the center of a given nonet, and they can be diagonalized by appropriate orthogonal transformations (for details see Appendix A). Besides the mixing inside the nonets there are other mixing terms, namely the last four terms of Eq. (9), which mix different nonets.

In order to eliminate the latter, one performs the following shifts of the (axial-)vector fields:

$$\begin{aligned}
f_{1N/S}^\mu & \longrightarrow f_{1N/S}^\mu + Z_{\eta_{N/S}}w_{f_{1N/S}}\partial^\mu\eta_{N/S}, \quad a_1^{\mu\pm,0} \longrightarrow a_1^{\mu\pm,0} + Z_\pi w_{a_1}\partial^\mu\pi^{\pm,0}, \\
K_1^{\mu\pm,0,\bar{0}} & \longrightarrow K_1^{\mu\pm,0,\bar{0}} + Z_K w_{K_1}\partial^\mu K^{\pm,0,\bar{0}}, \quad K^{*\mu\pm,0,\bar{0}} \longrightarrow K^{*\mu\pm,0,\bar{0}} + Z_{K^*} w_{K^*}\partial^\mu K_0^{*\pm,0,\bar{0}}.
\end{aligned} \tag{10}$$

These shifts produce additional kinetic terms for the pseudoscalar fields. In order to retain the canonical normalization for the latter, one has to introduce wavefunction renormalization constants,

$$\pi^{\pm,0} \rightarrow Z_\pi\pi^{\pm,0}, \quad K^{\pm,0,\bar{0}} \rightarrow Z_K K^{\pm,0,\bar{0}}, \quad \eta_{N/S} \rightarrow Z_{\eta_{N/S}}\eta_{N/S}, \quad K^{*\mu\pm,0,\bar{0}} \rightarrow Z_{K^*} K^{*\mu\pm,0,\bar{0}}. \tag{11}$$

For the sake of simplicity we have grouped together the isotriplet states with the notation  $\pi^{\pm,0}$ ,  $a_1^{\mu\pm,0}$  and the isodoublet states with the notation  $K^{\pm,0,\bar{0}}$ ,  $K^{*\mu\pm,0,\bar{0}}$ , where  $\bar{0}$  refers to  $\bar{K}^0$ . The coefficients  $w_i$  and  $Z_i$  are determined in order to eliminate the last four mixing terms in Eq. (9) and to obtain the canonical normalization of the  $\pi$ ,  $\eta_N$ ,  $\eta_S$ ,  $K$ , and  $K_0^*$  fields. After some straightforward calculation one finds the explicit expressions:

$$w_{f_{1N}} = w_{a_1} = \frac{g_1\phi_N}{m_{a_1}^2}, \quad w_{f_{1S}} = \frac{\sqrt{2}g_1\phi_S}{m_{f_{1S}}^2}, \quad w_{K^*} = \frac{ig_1(\phi_N - \sqrt{2}\phi_S)}{2m_{K^*}^2}, \quad w_{K_1} = \frac{g_1(\phi_N + \sqrt{2}\phi_S)}{2m_{K_1}^2}, \tag{12}$$

$$Z_\pi = Z_{\eta_N} = \frac{m_{a_1}}{\sqrt{m_{a_1}^2 - g_1^2\phi_N^2}}, \quad Z_K = \frac{2m_{K_1}}{\sqrt{4m_{K_1}^2 - g_1^2(\phi_N + \sqrt{2}\phi_S)^2}}, \tag{13}$$

$$Z_{\eta_S} = \frac{m_{f_{1S}}}{\sqrt{m_{f_{1S}}^2 - 2g_1^2\phi_S^2}}, \quad Z_{K^*} = \frac{2m_{K^*}}{\sqrt{4m_{K^*}^2 - g_1^2(\phi_N - \sqrt{2}\phi_S)^2}}. \tag{14}$$

It can be seen from the expressions of the wavefunction renormalization constants that they are always larger than one. Finally, using the explicit expressions found in Appendix A the tree-level (squared) masses for the different nonets are obtained as follows:

$$m_\pi^2 = Z_\pi^2 \left[ m_0^2 + \left( \lambda_1 + \frac{\lambda_2}{2} \right) \phi_N^2 + \lambda_1 \phi_S^2 \right] \equiv \frac{Z_\pi^2 h_{0N}}{\phi_N}, \tag{15}$$

$$m_K^2 = Z_K^2 \left[ m_0^2 + \left( \lambda_1 + \frac{\lambda_2}{2} \right) \phi_N^2 - \frac{\lambda_2}{\sqrt{2}} \phi_N \phi_S + (\lambda_1 + \lambda_2) \phi_S^2 \right], \tag{16}$$

$$m_{\eta_N}^2 = Z_\pi^2 \left[ m_0^2 + \left( \lambda_1 + \frac{\lambda_2}{2} \right) \phi_N^2 + \lambda_1 \phi_S^2 + c_1 \phi_N^2 \phi_S^2 \right] \equiv Z_\pi^2 \left( \frac{h_{0N}}{\phi_N} + c_1 \phi_N^2 \phi_S^2 \right), \tag{17}$$

$$m_{\eta_S}^2 = Z_{\eta_S}^2 \left[ m_0^2 + \lambda_1 \phi_N^2 + (\lambda_1 + \lambda_2) \phi_S^2 + \frac{c_1}{4} \phi_N^4 \right] \equiv Z_{\eta_S}^2 \left( \frac{h_{0S}}{\phi_S} + \frac{c_1}{4} \phi_N^4 \right), \tag{18}$$

$$m_{\eta_{NS}}^2 = Z_\pi Z_{\pi_S} \frac{c_1}{2} \phi_N^3 \phi_S, \tag{19}$$

are the (squared) pseudoscalar masses, while

$$m_{a_0}^2 = m_0^2 + \left( \lambda_1 + \frac{3}{2}\lambda_2 \right) \phi_N^2 + \lambda_1 \phi_S^2, \tag{20}$$

$$m_{K_0^*}^2 = Z_{K_0^*}^2 \left[ m_0^2 + \left( \lambda_1 + \frac{\lambda_2}{2} \right) \phi_N^2 + \frac{\lambda_2}{\sqrt{2}} \phi_N \phi_S + (\lambda_1 + \lambda_2) \phi_S^2 \right], \tag{21}$$



$$m_{\sigma_N}^2 = m_0^2 + 3 \left( \lambda_1 + \frac{\lambda_2}{2} \right) \phi_N^2 + \lambda_1 \phi_S^2, \quad (22)$$

$$m_{\sigma_S}^2 = m_0^2 + \lambda_1 \phi_N^2 + 3 (\lambda_1 + \lambda_2) \phi_S^2, \quad (23)$$

$$m_{\sigma_{NS}}^2 = 2\lambda_1 \phi_N \phi_S, \quad (24)$$

are the (squared) scalar masses. The quantities  $m_{\pi_{NS}}^2$  and  $m_{\sigma_{NS}}^2$  are mixing terms in the non-strange–strange sector. These mixings can be removed by orthogonal transformations, and the resulting mass eigenstates are found to be

$$m_{f_0^H/f_0^L}^2 = \frac{1}{2} \left[ m_{\sigma_N}^2 + m_{\sigma_S}^2 \pm \sqrt{(m_{\sigma_N}^2 - m_{\sigma_S}^2)^2 + 4m_{\sigma_{NS}}^4} \right], \quad (25)$$

$$m_{\eta'/\eta}^2 = \frac{1}{2} \left[ m_{\eta_N}^2 + m_{\eta_S}^2 \pm \sqrt{(m_{\eta_N}^2 - m_{\eta_S}^2)^2 + 4m_{\eta_{NS}}^4} \right]. \quad (26)$$

Moreover, the (squared) vector masses are given by

$$m_\rho^2 = m_1^2 + \frac{1}{2}(h_1 + h_2 + h_3)\phi_N^2 + \frac{h_1}{2}\phi_S^2 + 2\delta_N, \quad (27)$$

$$m_{K^*}^2 = m_1^2 + \frac{1}{4}(g_1^2 + 2h_1 + h_2)\phi_N^2 + \frac{1}{\sqrt{2}}\phi_N\phi_S(h_3 - g_1^2) + \frac{1}{2}(g_1^2 + h_1 + h_2)\phi_S^2 + \delta_N + \delta_S, \quad (28)$$

$$m_{\omega_N}^2 = m_\rho^2, \quad (29)$$

$$m_{\omega_S}^2 = m_1^2 + \frac{h_1}{2}\phi_N^2 + \left( \frac{h_1}{2} + h_2 + h_3 \right) \phi_S^2 + 2\delta_S, \quad (30)$$

while the (squared) axial-vector meson masses are

$$m_{a_1}^2 = m_1^2 + \frac{1}{2}(2g_1^2 + h_1 + h_2 - h_3)\phi_N^2 + \frac{h_1}{2}\phi_S^2 + 2\delta_N, \quad (31)$$

$$m_{K_1}^2 = m_1^2 + \frac{1}{4}(g_1^2 + 2h_1 + h_2)\phi_N^2 - \frac{1}{\sqrt{2}}\phi_N\phi_S(h_3 - g_1^2) + \frac{1}{2}(g_1^2 + h_1 + h_2)\phi_S^2 + \delta_N + \delta_S, \quad (32)$$

$$m_{f_{1N}}^2 = m_{a_1}^2, \quad (33)$$

$$m_{f_{1S}}^2 = m_1^2 + \frac{h_1}{2}\phi_N^2 + \left( 2g_1^2 + \frac{h_1}{2} + h_2 - h_3 \right) \phi_S^2 + 2\delta_S. \quad (34)$$

It is interesting to note that in case of vectors and axial-vectors there are no mixings in the non-strange–strange sector.

### C. Parameters

The Lagrangian (1) contains 18 parameters:

$$m_0^2, m_1^2, c_1, \delta_N, \delta_S, g_1, g_2, g_3, g_4, g_5, g_6, h_{0N}, h_{0S}, h_1, h_2, h_3, \lambda_1, \lambda_2. \quad (35)$$

We make the following observations regarding the model parameters:

- The parameters  $h_{0N}$  and  $h_{0S}$  model the explicit breaking of the chiral symmetry (ESB) in the (pseudo)scalar sector via the term  $\text{Tr}[H(\Phi + \Phi^\dagger)]$ ; they are uniquely determined from the mass terms of the pion, Eq. (15), and of  $\eta_S$ , the strange part of the  $\eta$  meson [see Eq. (18)], implying that the masses of  $\vec{\pi}$  and  $\eta_S$  are generated by ESB.
- The parameters  $\delta_N$  and  $\delta_S$  model the explicit symmetry breaking in the vector and axial-vector channels. The ESB arises from non-vanishing quark masses and therefore we employ the correspondence  $\delta_N \propto m_{u,d}^2$  and  $\delta_S \propto m_s^2$ . However, in the vector-meson mass term  $\text{Tr}[(m_1^2/2 + \Delta)(L_\mu^2 + R_\mu^2)]$  only the linear combinations  $m_1^2/2 + \delta_{N/S}$  appear. Therefore, it is possible to redefine  $m_1^2/2 \rightarrow m_1^2/2 - \delta_N$ . Then, only the combination  $\delta_S - \delta_N$  appears in the mass formulas. Only this difference will be determined by the fit of the (axial-)vector masses. Alternately, we may set  $\delta_N = 0$  from the beginning, and determine  $\delta_S$  from the fit.
- The parameters  $g_3, g_4, g_5$ , and  $g_6$  do not influence any of the decays to be discussed in this work and are therefore not considered in the fit.

Consequently, we are left with the following 13 parameters:

$$m_0^2, m_1^2, c_1, \delta_S, g_1, g_2, h_{0N}, h_{0S}, h_1, h_2, h_3, \lambda_1, \lambda_2. \quad (36)$$

Their large- $N_c$  dependence reads [15]:

$$m_0^2, m_1^2 \propto N_c^0; \quad h_{0N}, h_{0S} \propto N_c^{1/2}; \quad g_1, g_2 \propto N_c^{-1/2}; \quad \lambda_2, h_2, h_3 \propto N_c^{-1}; \quad \lambda_1, h_1 \propto N_c^{-2}; \quad c_1 \propto N_c^{-3}. \quad (37)$$

We recall that a non-suppressed  $n$ -meson interaction vertex scales as  $N_c^{-(n-2)/2}$  [30]. In this respect the parameters  $h_1$  and  $\lambda_1$  are suppressed in the large- $N_c$  limit: in fact, they scale as  $N_c^{-2}$  and not as  $N_c^{-1}$ . Similarly, the axial-anomaly parameter  $c_1$  scales as  $N_c^{-3}$  and not as  $N_c^{-2}$ . Note that the large- $N_c$  behaviour of the model parameters (37) implies that the decay widths (the formulas for which are given in Appendix B) decrease with increasing  $N_c$ . For this reason, as already mentioned, the states in our model represent  $\bar{q}q$  states.

### III. FIT: RESULTS AND CONSEQUENCES

#### A. Input and constraints for the fit

Let us now turn to the fit procedure for the parameters discussed above. Our fit aims to ascertain whether (i) it is possible to find a fit containing masses and decay widths for (pseudo)scalar and (axial-)vector degrees of freedom present in our model and (ii) which physical scalar states are best described by the  $\bar{q}q$  states of our model.

All decays considered are two-particle decays, thus they can be calculated from the appropriate tree-level terms of the Lagrangian (1) after applying the necessary field shifts (10) and orthogonal transformations (A11). If the decaying particle is denoted by  $A$  and the decay products by  $B$  and  $C$ , respectively, the decay width reads

$$\Gamma_{A \rightarrow BC} = \mathcal{I} \frac{|\mathbf{k}|}{8\pi m_A^2} |\mathcal{M}_{A \rightarrow BC}|^2, \quad (38)$$

where  $\mathbf{k}$  is the three-momentum of one of the resulting particles in the rest frame of  $A$  and  $\mathcal{M}_{A \rightarrow BC}$  is the transition matrix element (decay amplitude). Moreover,  $\mathcal{I}$  refers to the so-called isospin factor, which is the number of sub-channels in a given decay channel (for instance, if  $B = C = K$  the  $A \rightarrow KK$  decay can have two sub-channels, namely  $K^+K^-$  and  $\bar{K}^0K^0$ , which results in  $\mathcal{I} = 2$ ). Equation (38) will be used to calculate all decay widths entering the fit; details of the calculations can be found in Appendix B. Moreover, when identical particles emerge in the final state, the usual symmetry factors are included.

A brief remark is necessary regarding the errors used in the fit. The fit will contain input information from experimental data regarding both (axial-)vector and (pseudo)scalar states. The data are very precise for some of the resonances described by our model. For example, the mass of the  $\phi(1020)$  resonance (our  $\omega_S$  state) is known with 0.002% accuracy. Our model does not aim to describe hadron vacuum phenomenology with this extreme precision. The reason is simple: already isospin-breaking effects in the physical hadron mass spectrum are of the order of 5% [for instance the difference between the charged and neutral pion masses, or the masses of the  $a_1(1260)$  and the  $f_1(1285)$ ], but are completely neglected in our model. We therefore artificially increase the experimental errors to 5%, if the actual error is smaller, or we use the experimental values, if the error is larger than 5%.

Let us now discuss the input information for our fit. We do this separately for mesons of different spin [central values from PDG [1] unless otherwise stated, errors according to the above discussion]:

- *Weak decay constants:* We use

$$f_\pi = (92.2 \pm 4.6) \text{ MeV}, \quad f_K = (155.6/\sqrt{2} \pm 5.5) \text{ MeV}. \quad (39)$$

The following formulas relate the decay constants to the vacuum condensates:  $f_\pi = \phi_N/Z_\pi$  and  $f_K = (\sqrt{2}\phi_S + \phi_N)/(2Z_K)$ .

- *Pseudoscalar mesons:* We use

$$m_\pi = (138 \pm 6.9) \text{ MeV}, \quad m_K = (495.6 \pm 24.8) \text{ MeV}, \\ m_\eta = (547.9 \pm 27.4) \text{ MeV}, \quad m_{\eta'} = (957.8 \pm 47.9) \text{ MeV}. \quad (40)$$

with the following notes: (i)  $m_\pi$  and  $m_K$  represent isospin-averaged values; (ii) the relatively large error values come from the criterion  $\max(5\%, \text{experimental error})$ , discussed above.



- *Vector mesons:* We use

$$\begin{aligned} m_\rho &= (775.5 \pm 38.8) \text{ MeV}, \quad m_{K^*} = (893.8 \pm 44.7) \text{ MeV}, \quad m_\phi = (1019.5 \pm 51) \text{ MeV}, \\ \Gamma_{\rho \rightarrow \pi\pi} &= (149.1 \pm 7.4) \text{ MeV}, \quad \Gamma_{K^* \rightarrow K\pi} = (46.2 \pm 2.3) \text{ MeV}, \quad \Gamma_{\phi \rightarrow KK} = (3.54 \pm 0.178) \text{ MeV}, \end{aligned} \quad (41)$$

with the following notes: (i) we use the isospin-averaged value for  $m_{K^*}$ ; (ii) in case of the  $\phi$  decay we use the physical mass values in the kinematic factor  $k_K^3 = (m_\phi^2 - 4m_K^2)^{3/2}$  due to the proximity of  $m_{\phi(1020)}$  to the  $\bar{K}K$  threshold (in order to eliminate phase-space effects).

- *Axial-vector mesons:* We use

$$\begin{aligned} m_{a_1} &= (1230 \pm 61.5) \text{ MeV}, \quad m_{f_1(1420)} = (1426.4 \pm 71.3) \text{ MeV}, \\ \Gamma_{a_1 \rightarrow \rho\pi} &= (425 \pm 175) \text{ MeV}, \quad \Gamma_{a_1 \rightarrow \pi\gamma} = (0.640 \pm 0.250) \text{ MeV}, \quad \Gamma_{f_1(1420) \rightarrow K^*K} = (43.9 \pm 2.2) \text{ MeV}, \end{aligned} \quad (42)$$

with the following notes: (i) the value of  $\Gamma_{a_1 \rightarrow \rho\pi}$  is not precisely known; there are experimental data suggesting this decay channel to be dominant for  $a_1$  [31] and thus we estimate the possible range for  $\Gamma_{a_1 \rightarrow \rho\pi}$  from the interval for the full  $a_1$  decay width [= (250 – 600) MeV]; (ii) according to PDG the channel  $f_1(1420) \rightarrow K^*K$  is dominant within the channel  $f_1(1420) \rightarrow KK\pi$ , with the latter the overall dominant decay mode for  $f_1(1420)$ ; we have assumed  $\Gamma_{f_1(1420) \rightarrow KK\pi}$  to be equal to the full decay width  $\Gamma_{f_1(1420)} = (54.9 \pm 2.6) \text{ MeV}$  and determined  $\Gamma_{f_1(1420) \rightarrow K^*K}$  using an averaged branching ratio  $\Gamma_{f_1(1420) \rightarrow K^*K} / \Gamma_{f_1(1420) \rightarrow KK\pi} = (0.8 \pm 0.09)$  from Refs. [32] and [33].

- *Isotriplet and isodoublet scalar mesons:* The observables from Eqs. (40) – (42) will be used with any of the  $a_0$ - $K_0^*$  combinations [ $a_0(980)/K_0^*(800)$ ,  $a_0(980)/K_0^*(1430)$ ,  $a_0(1450)/K_0^*(800)$ ,  $a_0(1450)/K_0^*(1430)$ ], where the data to be used are as follows:

$$\begin{aligned} m_{a_0(980)} &= (980 \pm 49) \text{ MeV}, \quad m_{a_0(1450)} = (1474 \pm 74) \text{ MeV}, \\ m_{K_0^*(800)} &= (676 \pm 40) \text{ MeV}, \quad m_{K_0^*(1430)} = (1425 \pm 71) \text{ MeV}, \\ \Gamma_{a_0(1450)} &= (265 \pm 13.3) \text{ MeV}, \quad \Gamma_{K_0^*(800) \rightarrow K\pi} = (548 \pm 27.4) \text{ MeV}, \quad \Gamma_{K_0^*(1430) \rightarrow K\pi} = (270 \pm 80) \text{ MeV}, \\ |\mathcal{M}_{a_0(980) \rightarrow KK}| &= (3590 \pm 440) \text{ MeV} [34], \quad |\mathcal{M}_{a_0(980) \rightarrow \eta\pi}| = (3300 \pm 166.5) \text{ MeV} [34]. \end{aligned} \quad (43)$$

We note the following: (i) the interpretation of  $K_0^*(800)$  as a particle is controversial; we will nonetheless include it into our fits in accordance with the conclusions of Ref. [4]; (ii) as in the case of  $\phi(1020)$ , we will use the decay amplitudes rather than the decay widths for the processes  $a_0(980) \rightarrow KK$  and  $a_0(980) \rightarrow \eta\pi$  due to the proximity of  $a_0(980)$  to the  $\bar{K}K$  threshold. The reader may find it somewhat surprising that we are considering pairs of states above [ $a_0(1450)/K_0^*(1430)$ ] 1 GeV, below [ $a_0(980)/K_0^*(800)$ ] 1 GeV, as well as "mixed" pairs  $a_0(980)/K_0^*(1430)$ ,  $a_0(1450)/K_0^*(800)$ . The reason to consider also mixed pairs is that we want to avoid any kind of prejudice in the assignment of our  $\bar{q}q$  states, and thus explore all possibilities. Nonetheless, these mixed pairs as well as the pair below 1 GeV will be disfavored by our analysis (see below). Mass formulas used in the fit are presented in Eqs. (15) – (18), (20) – (23), and (25) – (34), whereas formulas for the decay widths are given in Appendix B.

Moreover, the fit shall be constrained by the following conditions:

- $Z_\pi, Z_K, Z_{\eta_S}, Z_{K_0^*} > 1$ , due to Eqs. (13) – (14).
- $m_{\eta_N} < m_{\eta_S}$  and  $m_{\sigma_N} < m_{\sigma_S}$ , i.e., pure non-strange states should be lighter than pure strange states.
- $m_0^2 < 0$ . This is a necessary condition for the spontaneous breaking of chiral symmetry.
- $\lambda_2 > 0$  and  $\lambda_1 > -\lambda_2/2$ . This is necessary for the potential in the Lagrangian (1) to be bounded from below.
- $m_1^2 \geq 0$ . The reason is that otherwise (i) an instability of the vacuum in the physical  $\rho$  direction would occur [see the Lagrangian (1)] and (ii)  $m_\rho$  and  $m_{a_1}$  would become imaginary in the chiral transition, i.e., once the condensates vanish [see Eqs. (27) and (31)].
- $m_1 \leq m_\rho$  as otherwise  $(h_1 + h_2 + h_3)\phi_N^2/2 + h_1\phi_S^2/2$  in the  $\rho$  mass term (27) would be negative; this would imply that spontaneous chiral symmetry breaking decreases the  $\rho$  mass. This is clearly unnatural because the breaking of chiral symmetry generates a sizable constituent mass for the light quarks, which is expected to positively contribute to the meson masses. This positive contribution is a feature of all known models (such as the Nambu–Jona-Lasinio model and constituent quark approaches). Indeed, in an important class of hadronic models [see Ref. [35] and refs. therein] the only and obviously positive contribution to the  $\rho$  mass is proportional to  $\phi^2$  (i.e.,  $m_1 = 0$ ).

Before discussing the results of the fit, it is important to stress which particles have not been included in the list above and why. Namely, we have omitted experimental information about the scalar-isoscalar and the axial-kaon states.

(i) Scalar-isoscalar states are not included in the fit because our theoretical approach is incomplete. There are three scalar states above 1 GeV, but at most two can be described within our model. Various approaches have shown [26] that a minimal model in this energy region requires three fields, where the additional state is the scalar glueball. Below 1 GeV the resonance  $f_0(500)$  is still poorly known and  $f_0(980)$  is distorted by the nearby  $\bar{K}K$  threshold.

(ii) The axial-kaon state  $K_1$  is not included in the fit for a similar reason: the kaonic states from our axial-vector nonet  $1^{++}$  mix with the kaonic states of the nonet  $1^{+-}$  which is not part of our Lagrangian. (This is possible, because charge conjugation is not a well-defined quantum number for kaons.) Also in this case we cannot assign our theoretical axial kaon field to a specific resonance [in PDG there are two axial kaons with masses  $K_1(1270)$  and  $K_1(1400)$ ], because the mixing is expected to be large, see e.g. Ref. [36].

## B. Results of the fit

The experimental quantities discussed in the previous subsection do not depend on all 13 parameters of Eq. (36), but on  $m_0, \lambda_1, m_1, h_1$  only through the two combinations

$$C_1 = m_0^2 + \lambda_1 (\phi_N^2 + \phi_S^2) \ , \quad C_2 = m_1^2 + \frac{h_1}{2} (\phi_N^2 + \phi_S^2) \ . \quad (44)$$

Moreover, instead of the parameters  $h_{0N}$  and  $h_{0S}$  we use the condensates  $\phi_N$  and  $\phi_S$ : this is equivalent, as  $h_{0N}, h_{0S}$  are completely determined by the masses of pion and  $\eta_S$ , cf. Eqs. (15), (18). Summarizing, we have the following eleven parameters entering the fit:

$$C_1, C_2, c_1, \delta_S, g_1, g_2, \phi_N, \phi_S, h_2, h_3, \lambda_2. \quad (45)$$

We perform our fit using the experimental values for the 17 quantities given in Eqs. (39) – (42). In addition, we use the experimental values for the pairs  $a_0(980) / K_0^*(800)$ ,  $a_0(980) / K_0^*(1430)$ ,  $a_0(1450) / K_0^*(800)$ ,  $a_0(1450) / K_0^*(1430)$  given in Eq. (43). These are four additional experimental quantities [except in the case of  $a_0(980)$ , where we use the two values for the decay amplitudes instead of one value for the decay width]. In total, we therefore fit 21 (22) experimental quantities to the eleven parameters given in Eq. (45). The corresponding values of  $\chi^2$  are listed in Table I. We see that the combination  $a_0(1450)/K_0^*(1430)$  gives the best value for  $\chi^2$ . In fact, with a  $\chi_{red}^2$  of 1.23 this fit is remarkably good, meaning that all physical quantities are reproduced within an average error of 5%. In the framework of an effective model for the strong interaction, which neglects isospin-breaking effects and describes all mesonic resonances up to about 1.8 GeV, we perceive this to be a remarkable achievement.

Note that also the fit with the pair  $a_0(980) / K_0^*(1430)$  has a  $\chi_{red}^2$  of the same order of magnitude as the one with the pair  $a_0(1450) / K_0^*(1430)$ . Nevertheless, we will disregard this fit for two reasons. Firstly, the  $\chi_{red}^2$  for the pair  $a_0(980) / K_0^*(1430)$  is, although rather small, still larger by a factor of about two than the  $\chi_{red}^2$  for the pair  $a_0(1450) / K_0^*(1430)$ . Secondly, and more importantly, the fit with the pair  $a_0(980) / K_0^*(1430)$  produces a scalar-kaon mass of  $m_{K_0^*} = 1146$  MeV, which cannot be assigned to any physical resonance as it is much larger than  $m_{K_0^*(800)}$  and much smaller than  $m_{K_0^*(1430)}$ . Note that this problem is also present in NJL models with mixing between scalar mesons below and above 1 GeV [37].

Pair	$\chi^2$	$\chi_{red}^2$
$a_0(1450)/K_0^*(1430)$	12.33	1.23
$a_0(980)/K_0^*(800)$	129.36	11.76
$a_0(980)/K_0^*(1430)$	22.00	2.00
$a_0(1450)/K_0^*(800)$	242.27	24.23

TABLE I: Isotriplet and isodoublet scalar pairs and the corresponding values of the total  $\chi^2$  and the reduced  $\chi_{red}^2 = \chi^2/N_{dof}$ , where  $N_{dof}$  is the difference between the number of experimental quantities and the number of fit parameters (10 for the first and fourth rows and 11 for the second and third rows).

The detailed comparison of theory with data for the pair  $a_0(1450) / K_0^*(1430)$  is presented in Table II where the theoretical errors are also shown. Errors for the model parameters ( $\delta p_i$ ) are calculated as the inverse square roots of the eigenvalues of the Hessian matrix obtained from  $\chi^2(p_j)$ , where  $p_j$  denotes elements of the parameter set (45) and

$i = 1, 2, \dots, 11$ . Theoretical errors  $\Delta O_i$  for each observable  $O_i$  (mass, decay width) can be calculated according to the following formula:

$$\Delta O_i = \sqrt{\sum_{j=1}^n \left( \left. \frac{\partial O_i}{\partial p_j} \right|_{\text{at fit value of } O_i} \delta p_j \right)^2}. \quad (46)$$

The remarkable agreement of our results with experimental data in the (pseudo)scalar and (axial-)vector sectors can now be explicitly seen for the various quantities shown in Table II. We thus conclude that the quark-antiquark state  $a_0$  and  $K_0^*$  should be assigned to the resonances  $a_0(1450)$  and  $K_0^*(1450)$ . Moreover, also the axial-vector resonances are well described by the fit: we therefore also interpret them as predominantly quark-antiquark states. In Table III we finally present the values of the parameters of the model together with their errors as obtained from our best fit.

Observable	Fit [MeV]	Experiment [MeV]
$f_\pi$	$96.3 \pm 0.7$	$92.2 \pm 4.6$
$f_K$	$106.9 \pm 0.6$	$110.4 \pm 5.5$
$m_\pi$	$141.0 \pm 5.8$	$137.3 \pm 6.9$
$m_K$	$485.6 \pm 3.0$	$495.6 \pm 24.8$
$m_\eta$	$509.4 \pm 3.0$	$547.9 \pm 27.4$
$m_{\eta'}$	$962.5 \pm 5.6$	$957.8 \pm 47.9$
$m_\rho$	$783.1 \pm 7.0$	$775.5 \pm 38.8$
$m_{K^*}$	$885.1 \pm 6.3$	$893.8 \pm 44.7$
$m_\phi$	$975.1 \pm 6.4$	$1019.5 \pm 51.0$
$m_{a_1}$	$1186 \pm 6$	$1230 \pm 62$
$m_{f_1(1420)}$	$1372.5 \pm 5.3$	$1426.4 \pm 71.3$
$m_{a_0}$	$1363 \pm 1$	$1474 \pm 74$
$m_{K_0^*}$	$1450 \pm 1$	$1425 \pm 71$
$\Gamma_{\rho \rightarrow \pi\pi}$	$160.9 \pm 4.4$	$149.1 \pm 7.4$
$\Gamma_{K^* \rightarrow K\pi}$	$44.6 \pm 1.9$	$46.2 \pm 2.3$
$\Gamma_{\phi \rightarrow \bar{K}K}$	$3.34 \pm 0.14$	$3.54 \pm 0.18$
$\Gamma_{a_1 \rightarrow \rho\pi}$	$549 \pm 43$	$425 \pm 175$
$\Gamma_{a_1 \rightarrow \pi\gamma}$	$0.66 \pm 0.01$	$0.64 \pm 0.25$
$\Gamma_{f_1(1420) \rightarrow K^*K}$	$44.6 \pm 39.9$	$43.9 \pm 2.2$
$\Gamma_{a_0}$	$266 \pm 12$	$265 \pm 13$
$\Gamma_{K_0^* \rightarrow K\pi}$	$285 \pm 12$	$270 \pm 80$

TABLE II: Best-fit results for masses and decay widths compared with experiment.

Parameter	Value
$C_1$ [GeV <sup>2</sup> ]	$-0.9183 \pm 0.0006$
$C_2$ [GeV <sup>2</sup> ]	$0.4135 \pm 0.0147$
$c_1$ [GeV <sup>-2</sup> ]	$450.5420 \pm 7.0339$
$\delta_S$ [GeV <sup>2</sup> ]	$0.1511 \pm 0.0038$
$g_1$	$5.8433 \pm 0.0176$
$g_2$	$3.0250 \pm 0.2329$
$\phi_N$ [GeV]	$0.1646 \pm 0.0001$
$\phi_S$ [GeV]	$0.1262 \pm 0.0001$
$h_2$	$9.8796 \pm 0.6627$
$h_3$	$4.8667 \pm 0.0864$
$\lambda_2$	$68.2972 \pm 0.0435$

TABLE III: Parameters and their errors.

### C. Consequences of the fit

#### 1. Scalar-isoscalar mesons

We now turn to the scalar-isoscalar mesons. We shall discuss four different aspects:

(a) *Results in the large- $N_c$  limit ( $\lambda_1 = h_1 = 0$ ):*

From the fit in the previous section we cannot immediately obtain the masses of the scalar-isoscalar states of the model because their masses do not depend solely on the combination  $C_1 = m_0^2 + \lambda_1 (\phi_N^2 + \phi_S^2)$ , but separately on the parameters  $m_0^2$  and  $\lambda_1$ . Similarly, the decay rates of the scalar-isoscalar mesons do not depend solely on the combination  $C_2 = m_1^2 + \frac{h_1}{2} (\phi_N^2 + \phi_S^2)$ , but separately on the parameters  $m_1$  and  $h_1$ .

Interestingly, the parameters  $\lambda_1$  and  $h_1$  are large- $N_c$  suppressed. Setting them to zero we obtain a prediction for the masses and decay widths of the scalar-isoscalar states in the large- $N_c$  limit. Moreover, when  $\lambda_1 = h_1 = 0$ , there is also no mixing between  $\bar{n}n$  and  $\bar{s}s$  states:  $f_0^L \equiv \sigma_N = \bar{n}n \equiv (\bar{u}u + \bar{d}d)/\sqrt{2}$  and  $f_0^H \equiv \sigma_S \equiv \bar{s}s$ . Their masses read:

$$m_{f_0^L} = 1362.7 \text{ MeV}, \quad m_{f_0^H} = 1531.7 \text{ MeV}. \quad (47)$$

These masses clearly lie above 1 GeV. Comparing these values to the experimental values for the three isoscalar states above 1 GeV [1]

$$m_{f_0(1370)} = (1350 \pm 150) \text{ MeV}, \quad m_{f_0(1500)} = (1505 \pm 75) \text{ MeV}, \quad m_{f_0(1710)} = (1720 \pm 86) \text{ MeV}, \quad (48)$$

we see that the mass of  $f_0^L$  is well compatible with that of the resonance  $f_0(1370)$ . [Note that  $m_{f_0(1370)}$  in Eq. (48) emulates the PDG mass interval (1200–1500) MeV.] The mass of  $f_0^H$  appears to be within that of  $f_0(1500)$ . Remember, though, that we artificially enlarged the experimental error to 5%; the actual error is only 6 MeV, and then the mass of  $f_0^H$  would be (just) outside the experimental error. Considering the decays of  $f_0^H$  we shall provide evidence that an assignment of  $f_0^H$  to  $f_0(1710)$  is also possible (and even rather likely).

For  $\lambda_1 = h_1 = 0$  the decay rates of  $f_0^{L,H}$  into  $\pi\pi$  and  $KK$  read

$$\Gamma_{f_0^L \rightarrow \pi\pi} = 520 \text{ MeV}, \quad \Gamma_{f_0^L \rightarrow KK} = 129 \text{ MeV}, \quad (49)$$

$$\Gamma_{f_0^H \rightarrow \pi\pi} = 0 \text{ MeV}, \quad \Gamma_{f_0^H \rightarrow KK} = 422 \text{ MeV}. \quad (50)$$

For the experimental values we quote the PDG values [1]:

$$\begin{aligned} \Gamma_{f_0(1370) \rightarrow \pi\pi} &= (250 \pm 100) \text{ MeV}, \quad \Gamma_{f_0(1370) \rightarrow KK} \lesssim \Gamma_{f_0(1370) \rightarrow \pi\pi}, \\ \Gamma_{f_0(1500) \rightarrow \pi\pi} &= (38 \pm 5) \text{ MeV}, \quad \Gamma_{f_0(1500) \rightarrow KK} = (9.4 \pm 2.3) \text{ MeV}, \\ \Gamma_{f_0(1710) \rightarrow \pi\pi} &= (29.3 \pm 6.5) \text{ MeV}, \quad \Gamma_{f_0(1710) \rightarrow KK} = (71.4 \pm 29.1) \text{ MeV}. \end{aligned} \quad (51)$$

Note that for the resonance  $f_0(1370)$  no branching ratios into  $\pi\pi$  and  $KK$  are reported in Ref. [1]. The value for  $\Gamma_{f_0(1370) \rightarrow \pi\pi}$  is our estimate from PDG and the review [7], whereas  $\Gamma_{f_0(1370) \rightarrow KK}$  is our estimate from results presented in Refs. [1, 38].

Our results for  $f_0^L$  are in agreement with the experimental decay widths of  $f_0(1370)$ . Our theoretical value for  $f_0^H \rightarrow KK$  turns out to be too large, while  $f_0^H \rightarrow \pi\pi$  vanishes, because  $f_0^H \equiv \sigma_S$  is a pure  $\bar{s}s$  state. Nevertheless, our model predicts the existence of a scalar-isoscalar state which decays predominantly into kaons; this is indeed the decay pattern shown by  $f_0(1710)$ . For these reasons we identify our state  $f_0^H$  as (predominantly)  $f_0(1710)$ . This identification will be further tested when  $\lambda_1 \neq 0$ ,  $h_1 \neq 0$  below.

It should be stressed that a quantitative study of the scalar-isoscalar system cannot be performed at present because our model contains only two states, while three resonances appear (this is also the reason why we did not include the scalar-isoscalar states into the fit). As many studies confirm [26], the mixing with the scalar glueball can be sizable: a reliable analysis of the scalar-isoscalar states can only be performed when taking the scalar glueball into account. The preliminary results in the  $N_f = 2$  sector [28] have indeed shown that the glueball and quarkonia degrees of freedom interact strongly and that the decay patterns are sizably influenced.

(b) *Results for  $\lambda_1 \neq 0$ ,  $h_1 \neq 0$ :*

A nonvanishing value of the parameter  $\lambda_1$  induces a mixing of the pure states  $\sigma_N$  and  $\sigma_S$ , see Eq. (24). Our

fit determines only the value of the linear combination  $C_1$  rather than the value of  $\lambda_1$ , see discussion before Eq. (45). Nonetheless, a range of values for  $\lambda_1$  can be estimated using the value of  $C_1$  from the fit and the condition  $m_0^2 < 0$ . Consequently, we also obtain a range of values for our isoscalar masses  $m_{f_0^{L,H}}$  and isoscalar decay widths. The masses vary in the following intervals:  $415 \text{ MeV} \leq m_{f_0^L} \leq 1460 \text{ MeV}$  and  $1480 \text{ MeV} \leq m_{f_0^H} \leq 1981 \text{ MeV}$ . Considering these mass values,  $f_0^L$  may correspond to either  $f_0(500)$ ,  $f_0(980)$ , or  $f_0(1370)$  and  $f_0^H$  may correspond to either  $f_0(1500)$  or  $f_0(1710)$  [1]. Therefore, a mere calculation of scalar masses does not allow us to assign the scalar states  $f_0^L$  and  $f_0^H$  to physical resonances. In order to resolve this ambiguity, we will calculate various decay widths of the states  $f_0^L$  and  $f_0^H$  and compare them to data [1].

The dependence of  $\Gamma_{f_0^{L,H} \rightarrow \pi\pi}$  on  $m_{f_0^{L,H}}$  is presented in Fig. 1.

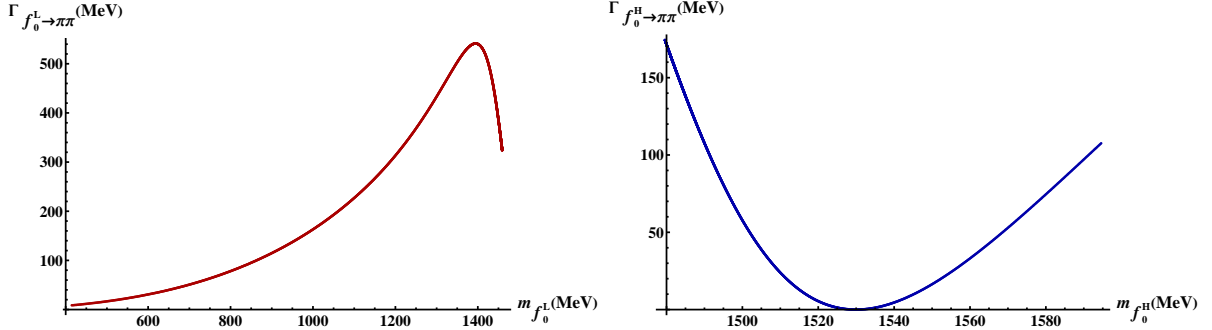


FIG. 1:  $\Gamma_{f_0^L \rightarrow \pi\pi}$  and  $\Gamma_{f_0^H \rightarrow \pi\pi}$  as functions of  $m_{f_0^L}$  and  $m_{f_0^H}$ , respectively.

The decay width  $\Gamma_{f_0^L \rightarrow \pi\pi}$  is consistent with the experimental range  $(250 \pm 100) \text{ MeV}$  for the  $f_0(1370) \rightarrow \pi\pi$  decay width, if  $1000 \text{ MeV} \lesssim m_{f_0^L} \leq 1460 \text{ MeV}$ . Other assignments can be excluded: (i) our  $f_0^L$  state cannot be assigned to  $f_0(500)$  as  $\Gamma_{f_0^L \rightarrow \pi\pi} \lesssim 20 \text{ MeV}$  in the (new) PDG mass range  $(400 - 550) \text{ MeV}$  for  $f_0(500)$ . Therefore, this strongly disfavors  $f_0(500)$  as a  $\bar{q}q$  state. (ii)  $\Gamma_{f_0^L \rightarrow \pi\pi}$  varies between approximately 130 MeV and 150 MeV in the mass interval of the  $f_0(980)$  resonance, i.e., the interval between 960 MeV and 980 MeV. The PDG result for the full decay width of the  $f_0(980)$  resonance is between 40 MeV and 100 MeV, with the  $\pi\pi$  channel being dominant [1], i.e., about a factor two smaller than our theoretical values. Consequently, the assignment of  $f_0^L$  to  $f_0(980)$  is also disfavored by our analysis. Therefore, (although there could still be mixing with other states) we assign our  $f_0^L$  state to  $f_0(1370)$ , thus supporting the interpretation of the latter state as a quarkonium. Although experimental data is not conclusive, we mention that the decay width  $\Gamma_{f_0^L \rightarrow KK}$  shown in Fig. 2 is consistent with the data.

In the mass interval  $1500 \text{ MeV} \lesssim m_{f_0^H} \lesssim 1560 \text{ MeV}$ , our results for the decay width  $\Gamma_{f_0^H \rightarrow \pi\pi}$  are consistent with experimental values for both  $f_0(1500)$  and  $f_0(1710)$ , see Eq. (51). However, the  $KK$  decay widths present us with a different conclusion: as already indicated in the case  $\lambda_1 = h_1 = 0$ , our  $f_0^H$  state decays much more abundantly into kaons than into pions, and the experimental data suggest only one physical resonance with the same feature:  $f_0(1710)$ . The other resonance  $f_0(1500)$  decays preferably into pions. Thus (and although the mass value for  $f_0^H$  is too small in the range where the decay width into pions agrees with the experimental value) we assign our  $f_0^H$  state to  $f_0(1710)$  and inspect in the following whether this assignment is justified by other data.

We observe from Fig. 2 that  $\Gamma_{f_0^H \rightarrow KK}$  rises rapidly and remains above the PDG result  $\Gamma_{f_0(1710) \rightarrow KK} = (71.4 \pm 29.1) \text{ MeV}$  in the entire mass interval  $m_{f_0^H} \gtrsim 1500 \text{ MeV}$ . Although the absolute value of the decay width is rather large, several ratios of decay widths can be described correctly by our fit, most notably the  $\pi\pi/KK$  decay ratio presented in Fig. 3.

Let us first discuss results for  $\Gamma_{f_0^L \rightarrow KK} / \Gamma_{f_0^L \rightarrow \pi\pi}$  (left panel of Fig. 3). We observe that the ratio varies between 0.49 for  $m_{f_0^L} = 1152 \text{ MeV}$  and 0 for  $m_{f_0^L} = 1444 \text{ MeV}$ . Experimental data regarding this ratio for  $f_0(1370)$  are unfortunately inconclusive,  $\Gamma_{f_0(1370) \rightarrow KK} / \Gamma_{f_0(1370) \rightarrow \pi\pi} = 0.08 \pm 0.08$  is quoted by the BESII Collaboration [39] and  $\Gamma_{f_0(1370) \rightarrow KK} / \Gamma_{f_0(1370) \rightarrow \pi\pi} = 0.91 \pm 0.20$  is the value given by the OBELIX Collaboration [40]; for the mass interval shown in Fig. 3, our result is most consistent with the value of the WA102 Collaboration [41],  $\Gamma_{f_0(1370) \rightarrow KK} / \Gamma_{f_0(1370) \rightarrow \pi\pi} = 0.46 \pm 0.15 \pm 0.11$ . The ambiguities in the experimental value of this ratio do not

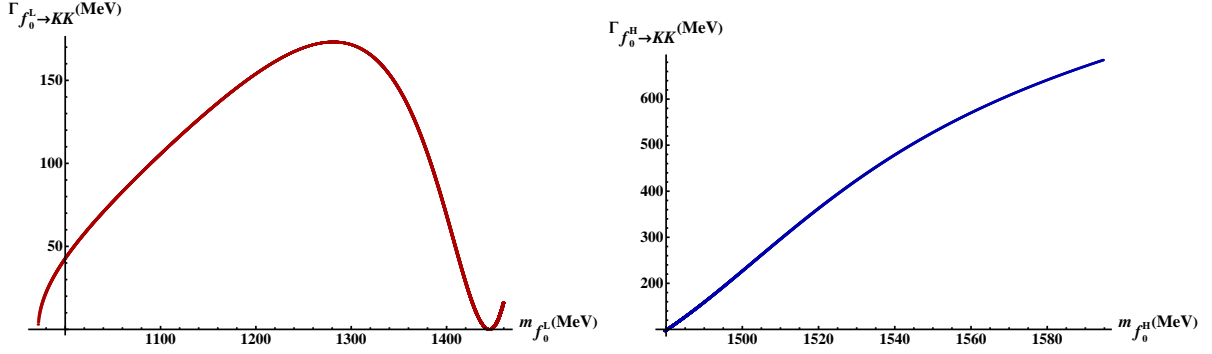


FIG. 2:  $\Gamma_{f_0^L \rightarrow KK}$  and  $\Gamma_{f_0^H \rightarrow KK}$  as functions of  $m_{f_0^L}$  and  $m_{f_0^H}$ , respectively.

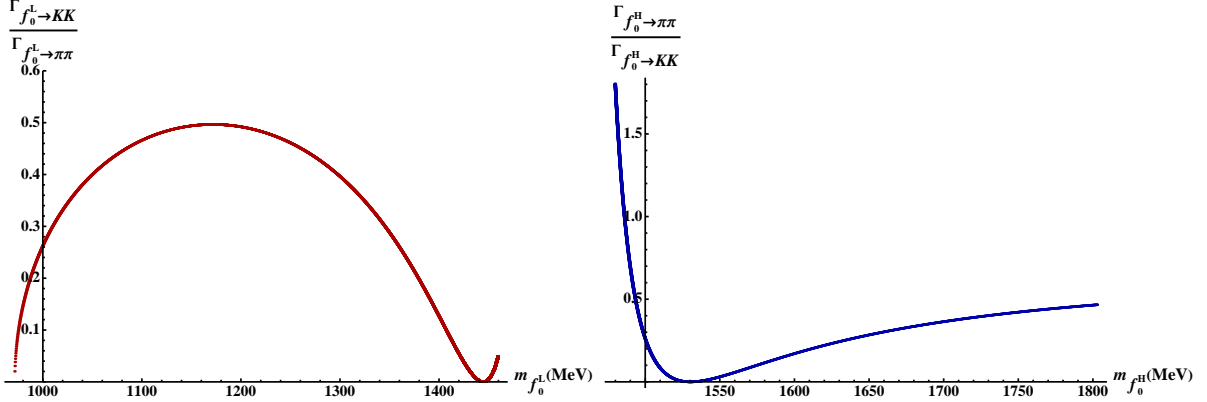


FIG. 3: Left panel: ratio  $\Gamma_{f_0^L \rightarrow KK} / \Gamma_{f_0^L \rightarrow \pi\pi}$  as function of  $m_{f_0^L}$ . Right panel: ratio  $\Gamma_{f_0^H \rightarrow \pi\pi} / \Gamma_{f_0^H \rightarrow KK}$  as function of  $m_{f_0^H}$ .

allow us to constrain our parameters, although our  $f_0^L$  state is compatible with the  $f_0(1370)$  data also for this particular case.

Let us now discuss results for  $\Gamma_{f_0^H \rightarrow \pi\pi} / \Gamma_{f_0^H \rightarrow KK}$  (right panel of Fig. 3). As already mentioned, the analysis of the two-pion decay did not allow for a definitive assignment of our  $f_0^H$  state, as it could correspond either to  $f_0(1500)$  or to  $f_0(1710)$ . The PDG data suggest the ratio  $\Gamma_{f_0(1500) \rightarrow KK} / \Gamma_{f_0(1500) \rightarrow \pi\pi} = 0.246 \pm 0.026$  or  $\Gamma_{f_0(1500) \rightarrow \pi\pi} / \Gamma_{f_0(1500) \rightarrow KK} = 4.065 \pm 0.430$ , respectively, whereas, for  $f_0(1710)$ , we use the WA102 ratio  $\Gamma_{f_0(1710) \rightarrow \pi\pi} / \Gamma_{f_0(1710) \rightarrow KK} = 0.2 \pm 0.06$  [41]. The latter is only marginally consistent with the one preferred by the PDG [ $\Gamma_{f_0(1710) \rightarrow \pi\pi}^{\text{PDG}} / \Gamma_{f_0(1710) \rightarrow KK}^{\text{PDG}} = 0.41^{+0.11}_{-0.17}$ ], originally published by the BESII Collaboration [42], that suffers from a large background in the  $\pi^+\pi^-$  channel (approximately 50%) and is therefore omitted from our considerations. We observe from the right panel of Fig. 3 that our ratio  $\Gamma_{f_0^H \rightarrow \pi\pi} / \Gamma_{f_0^H \rightarrow KK}$  never corresponds to the one experimentally determined for  $f_0(1500)$ . Although the ratio shows a strong increase near the left border of the mass interval, decreasing the mass beyond this border would violate the constraint  $m_0^2 < 0$ . Thus, the experimental value is out of reach. Conversely, our ratio describes exactly the value  $\Gamma_{f_0(1710) \rightarrow \pi\pi} / \Gamma_{f_0(1710) \rightarrow KK} = 0.2 \pm 0.06$  if  $m_{f_0^H} = 1502^{+2}_{-3}$  MeV. [We discard the second possibility  $m_{f_0^H} = 1611^{+27}_{-23}$  MeV as the decay width  $\Gamma_{f_0^H \rightarrow KK}$  would exceed 600 MeV, see Fig. 2, i.e., an order of magnitude larger than the experimental value, see Eq. (51).] This implies  $\lambda_1 = -4.1 \mp 0.7$  – the parameter  $\lambda_1$  remains close to the large- $N_c$  limit and is much smaller than  $\lambda_2$ , see Table III.

The contribution of the pure-strange state  $\sigma_S$  to  $f_0^H$  is then approximately 96%, as can be calculated from Eqs. (A12) – (A16). Thus our predominantly strange state  $f_0^H$  describes the  $\pi\pi/KK$  ratio of  $f_0(1710)$ , and not that of  $f_0(1500)$ , although the correct description requires that  $m_{f_0^H}$  corresponds to the mass value of  $f_0(1500)$ . The correct description of the decay ratio is an indication that our assignment of  $f_0^H$  to  $f_0(1710)$  is the correct one, whereas the fact that  $m_{f_0^H}$  is smaller than  $m_{f_0(1710)}$  indicates the necessity to include a third isoscalar



degree of freedom into our model: the glueball. As concluded from the  $N_f = 2$  version of our model [28], the  $f_0(1500)$  is predominantly a glueball state and thus the inclusion of the glueball state is expected to induce a level repulsion in the masses. This may shift the currently too small mass value of the predominantly strange quarkonium from approximately 1.5 GeV to 1.7 GeV, where it is experimentally found. Nonetheless, our results clearly demonstrate that scalar quarkonia are found above, rather than below, 1 GeV.

Until now we have only considered the case where one of our large- $N_c$  suppressed parameters ( $\lambda_1$ ) is nonzero. We have also investigated the influence of non-vanishing values for the other large- $N_c$  suppressed parameter,  $h_1$ , on the decay widths  $\Gamma_{f_0^{L,H} \rightarrow \pi\pi}$  and  $\Gamma_{f_0^{L,H} \rightarrow KK}$  [see Eqs. (B24), (B25), (B30), and (B31)]. Consistency with the large- $N_c$  deliberations requires us to keep  $h_1$  smaller than, or in the vicinity of,  $h_2$  and  $h_3$ , see Table III. We again observe that our ratio  $\Gamma_{f_0^H \rightarrow \pi\pi}/\Gamma_{f_0^H \rightarrow KK}$  never corresponds to the one of the  $f_0(1500)$  resonance whereas the ratio  $\Gamma_{f_0(1710) \rightarrow \pi\pi}/\Gamma_{f_0(1710) \rightarrow KK} = 0.2 \pm 0.06$  is correctly described if  $h_1 \sim -9$ . In this case,  $m_{f_0^H}$  rises to approximately 1540 MeV and is thus outside the PDG result  $m_{f_0(1500)} = (1505 \pm 6)$  MeV but still too small when compared to  $m_{f_0(1710)} = (1720 \pm 6)$  MeV. Thus, it is still necessary to include a glueball degree of freedom into our model. The assignment of  $f_0^H$  to  $f_0(1710)$  is further justified considering decays into  $\eta$  and  $\eta'$  mesons.

(c)  $\eta\eta$  and  $\eta\eta'$  decay channels for the scalar-isoscalar mesons:

PDG data suggest the following values of  $\eta\eta$  decay widths for  $f_0(1500)$  and  $f_0(1710)$  [1]

$$\Gamma_{f_0(1500) \rightarrow \eta\eta} = (5.56 \pm 1.34) \text{ MeV}, \quad \Gamma_{f_0(1710) \rightarrow \eta\eta} = (38.6 \pm 18.8) \text{ MeV}. \quad (52)$$

Our analysis always yields  $\Gamma_{f_0^H \rightarrow \eta\eta} > 20$  MeV; there is consequently no value of  $m_{f_0^H}$  at which  $\Gamma_{f_0^H \rightarrow \eta\eta}$  would be compatible with  $\Gamma_{f_0(1500) \rightarrow \eta\eta}$ . However,  $m_{f_0^H} = 1502_{-3}^{+2}$  MeV [obtained requiring  $\Gamma_{f_0^H \rightarrow \pi\pi}/\Gamma_{f_0^H \rightarrow KK} = \Gamma_{f_0(1710) \rightarrow \pi\pi}/\Gamma_{f_0(1710) \rightarrow KK}$ , i.e.,  $\lambda_1 = -4.1 \mp 0.7$  and  $h_1 = 0$ ] yields

$$\Gamma_{f_0^H \rightarrow \eta\eta} = 49.6_{-3.3}^{+4.1} \text{ MeV}, \quad (53)$$

(where the errors arise from the uncertainty in  $\lambda_1$  only), i.e., in agreement with the experimental value quoted in Eq. (52). The same parameter set also yields

$$\Gamma_{f_0^L \rightarrow \eta\eta} \simeq 33 \text{ MeV}; \quad (54)$$

purely a prediction, as  $\Gamma_{f_0(1370) \rightarrow \eta\eta}$  has not yet been measured.

The choice  $\lambda_1 = -4.1 \mp 0.7$ ,  $h_1 = 0$  also yields

$$\Gamma_{f_0^L \rightarrow \eta\eta}/\Gamma_{f_0^L \rightarrow KK} = 0.194_{-0.003}^{+0.002}, \quad \Gamma_{f_0^H \rightarrow \eta\eta}/\Gamma_{f_0^H \rightarrow KK} = 0.204_{-0.002}^{+0.001}, \quad (55)$$

$$\Gamma_{f_0^L \rightarrow \eta\eta}/\Gamma_{f_0^L \rightarrow \pi\pi} = 0.087_{-0.006}^{+0.005}, \quad \Gamma_{f_0^H \rightarrow \eta\eta}/\Gamma_{f_0^H \rightarrow \pi\pi} = 1.02_{+0.42}^{-0.23}, \quad (56)$$

whereas experimental data read

$$\Gamma_{f_0(1500) \rightarrow \eta\eta}/\Gamma_{f_0(1500) \rightarrow KK} = 0.59 \pm 0.12 \quad [1], \quad (57)$$

$$\Gamma_{f_0(1500) \rightarrow \eta\eta}/\Gamma_{f_0(1500) \rightarrow \pi\pi} = 0.145 \pm 0.027 \quad [1], \quad (58)$$

$$\Gamma_{f_0(1710) \rightarrow \eta\eta}/\Gamma_{f_0(1710) \rightarrow KK} = 0.48 \pm 0.15 \quad [43], \quad (59)$$

$$\Gamma_{f_0(1710) \rightarrow \eta\eta}/\Gamma_{f_0(1710) \rightarrow KK} = 0.46_{-0.38}^{+0.70} \quad [44], \quad (60)$$

$$\Gamma_{f_0(1710) \rightarrow \eta\eta}/\Gamma_{f_0(1710) \rightarrow \pi\pi} = 2.40 \pm 1.04 \quad [41, 43]. \quad (61)$$

In all cases, the  $f_0(1500)$  data are off by several standard deviations from our theoretical results, while there is good agreement with the results for  $f_0(1710)$ .

We have also considered decays involving the  $\eta'$  meson. Since the threshold for  $\eta\eta'$  decays is at approximately 1.5 GeV, it suffices to consider decays of  $f_0^H$  only. The value  $\lambda_1 = -4.1 \mp 0.7$  yields

$$\Gamma_{f_0^H \rightarrow \eta\eta'} = 12.7_{-1.4}^{+1.1} \text{ MeV}, \quad (62)$$

$$\Gamma_{f_0^H \rightarrow \eta\eta'}/\Gamma_{f_0^H \rightarrow \pi\pi} = 0.26_{+0.04}^{-0.03}, \quad (63)$$

$$\Gamma_{f_0^H \rightarrow \eta\eta'}/\Gamma_{f_0^H \rightarrow \eta\eta} = 0.26_{-0.05}^{+0.04}, \quad (64)$$

$$\Gamma_{f_0^H \rightarrow \eta\eta'}/\Gamma_{f_0^H \rightarrow KK} = 0.05 \pm 0.01, \quad (65)$$

whereas experimental data read

$$\Gamma_{f_0(1500) \rightarrow \eta\eta'} = (2.1 \pm 1.0) \text{ MeV} , \quad (66)$$

$$\Gamma_{f_0(1500) \rightarrow \eta\eta'} / \Gamma_{f_0(1500) \rightarrow \pi\pi} = 0.055 \pm 0.024 , \quad (67)$$

$$\Gamma_{f_0(1500) \rightarrow \eta\eta'} / \Gamma_{f_0(1500) \rightarrow \eta\eta} = 0.38 \pm 0.16 . \quad (68)$$

The decay ratio  $\Gamma_{f_0(1500) \rightarrow \eta\eta'} / \Gamma_{f_0(1500) \rightarrow KK}$  is unknown; there are also no data for the  $\eta\eta'$  decay channel of  $f_0(1710)$ . Still we observe that neither  $\Gamma_{f_0^H \rightarrow \eta\eta'}$  nor  $\Gamma_{f_0^H \rightarrow \eta\eta'} / \Gamma_{f_0^H \rightarrow \pi\pi}$  describe the corresponding experimental results for  $f_0(1500)$ . Indeed the only piece of experimental data regarding  $f_0(1500)$  that is described by our fit results is the one for the  $\eta\eta' / \eta\eta$  decay ratio. Nonetheless, all the other results regarding decay ratios obtained by our analysis clearly demonstrate the correspondence of our predominantly strange state  $f_0^H$  to  $f_0(1710)$ ;  $f_0^L$  was found to correspond to  $f_0(1370)$  already in the discussion of the  $\pi\pi$  decay channel. Consequently, we conclude that  $f_0(1370)$  and  $f_0(1710)$  are favoured as scalar  $\bar{q}q$  states. However, we also stress again that the mass of our  $f_0^H$  remains too low when compared to  $m_{f_0(1710)}$  due to a missing scalar-glueball state expected to shift  $m_{f_0^H}$  to  $m_{f_0(1710)}$  by level repulsion.

## 2. Mixing in the pseudoscalar-isoscalar sector

Our Lagrangian (1) implements the mixing of two pure pseudoscalar isosinglet states,  $\eta_N \equiv (\bar{u}u + \bar{d}d)/\sqrt{2}$  and  $\eta_S \equiv \bar{s}s$ . The mixing term is presented in Eq. (19). The mass terms for  $\eta_N$  (17) and  $\eta_S$  (18) are determined by our fit parameters presented in Table III. The same parameters also determine the  $\eta$ - $\eta'$  mixing angle as [see Eqs. (A12) – (A16)]

$$\theta_\eta = -44.6^\circ . \quad (69)$$

The result is close to ideal mixing.

## 3. The axial-vector kaon state $K_1$

The  $K_1$  state has not been assigned to a physical resonance and included into our fit, because the PDG listing suggest two distinct assignment candidates:  $K_1(1270)$  and  $K_1(1400)$ , expected to mix [36]. Therefore, mass and decay widths are left as predictions of this work.

There are three decay widths of the  $K_1$  state that can be calculated within our model at tree level:  $K_1 \rightarrow K^*\pi$ ,  $\rho K$ , and  $\omega K$ . They account for approximately 70% of the full  $K_1(1270)$  decay width and almost 100% of the full  $K_1(1400)$  decay width [1]. Using the parameter values stated in Table III it is possible to calculate the mass, Eq. (32), as well as the decay width [via the generic decay-width formula (B46)] of our  $K_1$  state. We obtain the following results:

$$m_{K_1} = 1282 \text{ MeV} , \quad \Gamma_{K_1 \rightarrow K^*\pi} = 205 \text{ MeV} , \quad \Gamma_{K_1 \rightarrow \rho K} = 44 \text{ MeV} , \quad \Gamma_{K_1 \rightarrow \omega K} = 15 \text{ MeV} . \quad (70)$$

The mass is within  $2\sigma$  of  $m_{K_1(1270)} = (1272 \pm 7) \text{ MeV}$  and thus rather close to the experimental result. However, the value of the full decay width is 264 MeV, while the PDG data read  $\Gamma_{K_1(1270)} = (90 \pm 20) \text{ MeV}$  and  $\Gamma_{K_1(1400)} = (174 \pm 13) \text{ MeV}$ . Our result is therefore approximately three times too large when compared to the data for  $K_1(1270)$  and approximately 50% too large when compared to the data for  $K_1(1400)$ ; errors have been omitted from the calculation. These results demonstrate the necessity to include a pseudovector  $I(J^{PC}) = 1(1^{+-})$  nonet into our model and implement its mixing with the already present axial-vector nonet. A possible mixing term between an axial-vector nonet  $A_1$  and a pseudovector nonet  $B_1$  reads

$$\text{Tr}(\Delta[A_1^\mu, B_{1\mu}]) , \quad (71)$$

with  $\Delta$  from Eq. (7). Various studies have indeed found the mixing to be non-negligible [36]; see also Ref. [45].

## 4. Branching ratios of $a_0(1450)$

As a consequence of our fit we can determine the branching ratio of the resonance  $a_0(1450)$  into  $KK$ ,  $\pi\eta$ , and  $\pi\eta'$ . We obtain the following values [see Eqs. (B1), (B2), and (B11)]:

$$\Gamma_{a_0 \rightarrow \pi\eta} = (115.4 \pm 6.2) \text{ MeV} , \quad \Gamma_{a_0 \rightarrow \eta'\pi} = (21.5 \pm 1.4) \text{ MeV} , \quad \Gamma_{a_0 \rightarrow KK} = (128.8 \pm 3.9) \text{ MeV} . \quad (72)$$

This leads to the following branching ratios:

$$\frac{\Gamma_{a_0 \rightarrow \eta' \pi}}{\Gamma_{a_0 \rightarrow \eta \pi}} = 0.19 \pm 0.02, \quad \frac{\Gamma_{a_0 \rightarrow KK}}{\Gamma_{a_0 \rightarrow \eta \pi}} = 1.12 \pm 0.07, \quad (73)$$

which should be compared with the experimental results [1]

$$\frac{\Gamma_{a_0(1450) \rightarrow \eta' \pi}}{\Gamma_{a_0(1450) \rightarrow \eta \pi}} = 0.35 \pm 0.16, \quad \frac{\Gamma_{a_0(1450) \rightarrow KK}}{\Gamma_{a_0(1450) \rightarrow \eta \pi}} = 0.88 \pm 0.23. \quad (74)$$

Our results are, within errors, consistent with the data.

### 5. Contributions to the mass of the $\rho$ meson

The mass of the  $\rho$  meson consists of three terms,

$$m_\rho^2 = m_1^2 + \frac{1}{2}(h_1 + h_2 + h_3)\phi_N^2 + \frac{h_1}{2}\phi_S^2. \quad (75)$$

The first term  $m_1^2$  is generated by the condensation of the dilaton field  $G$ ,  $m_1^2 \propto G_0^2$ . The second term is proportional to the chiral condensate  $\phi_N^2$ . In the large- $N_c$  limit the parameter  $h_1 = 0$  and we can determine the terms from the results of the fit:  $m_1 = 0.643$  GeV and  $\sqrt{(h_2 + h_3)/2}\phi_N = 0.447$  GeV. It turns out that the glueball-driven term is dominant.

It is interesting to note that models based on QCD sum rules [46] or Brown-Rho scaling [47] predict that  $m_\rho^k \propto \phi_N$ , where  $k = 2$  in the first and  $k = 3$  in the latter case. The order parameter for chiral symmetry breaking  $\phi_N$  decreases as a function of nuclear matter density  $n$ : for small densities it is well known (see, e.g. Ref. [46]) that

$$\frac{\phi_N|_n}{\phi_N|_{\text{vacuum}}} \approx 1 - 0.3 \frac{n}{n_0},$$

so these models predict a substantial decrease of  $m_\rho$  already at nuclear matter saturation density,  $n_0$ . In this respect, it is interesting to evaluate the value of the  $\rho$  mass at  $n_0$  according to Eq. (75). We assume that the glueball-driven term  $m_1^2$  does not vary. This can be motivated by considering that the glueball is massive and a substantial decrease in its mass may occur only at much higher density. Assuming that the linear dependence on the density of the condensate holds up to saturation density, we obtain that  $m_\rho$  decreases by about 70 MeV compared to 90 – 130 MeV predicted by Refs. [46, 47].

Obviously, these considerations are only qualitative but could represent the starting point of interesting studies of vector mesons in the medium, which is an important aim of several experimental investigations [48]. To this end, one has to calculate the behavior of the dilaton and chiral condensates in the medium within the same theoretical framework.

## IV. CONCLUSIONS

We have presented a linear sigma model with three quark flavors. The model implements the symmetries of QCD, the discrete  $CPT$  symmetry, the global chiral  $U(N_f)_L \times U(N_f)_R$  symmetry, and the breaking mechanisms of the latter: spontaneous (due to the chiral condensate), explicit (due to non-vanishing quark masses) as well as at the quantum level [the  $U(1)_A$  anomaly].

The model has been used to describe meson states up to energies of  $\sim 1.7$  GeV. This energy region exhibits numerous resonances, related by scattering reactions and decays. For this reason, a realistic model of QCD degrees of freedom in the mentioned energy region should describe as many of the resonances as possible. Thus, we have constructed a linear sigma model that contains scalar (two isoscalars,  $f_0^L$  and  $f_0^H$ , as well as an isotriplet,  $a_0$ , and two isodoublets,  $K_0^*$ ), pseudoscalar ( $\pi$ ,  $K$ ,  $\eta$ ,  $\eta'$ ), vector ( $\rho$ ,  $\omega$ ,  $K^*$ ,  $\phi$ ), and axial-vector [ $a_1$ ,  $K_1$ ,  $f_1(1285)$ ,  $f_1(1420)$ ] degrees of freedom. We have thus constructed a single model that contains (pseudo)scalars and (axial-)vectors both in the non-strange and strange channels. To our knowledge, this is the first time that such a comprehensive approach has been presented.

The model, dubbed extended Linear Sigma Model (eLSM), has allowed us to study the overall phenomenology of mesons and, in particular, to explore the nature of scalar and axial-vector resonances. In order to test our model we

have performed a global fit to 21 experimental quantities involving both the (pseudo)scalar and the (axial-)vector masses and decays. Due to mixing with the scalar glueball (not included here), we did not include the scalar-isoscalar resonances in the fit. Similarly, we have omitted the axial-vector resonance  $K_1$ , due to the fact that in reality a large mixing of two kaonic fields from the  $1^{++}$  and  $1^{+-}$  nonets takes place.

One of the central questions of our discussion has been the assignment of the scalar states: to this end we have tested the possible scenarios for the isotriplet and isodoublet scalar states by assigning our scalar fields  $a_0$  to  $a_0(980)$  or  $a_0(1450)$  and  $K_0^*$  to  $K_0^*(800)$  or  $K_0^*(1430)$ . The outcome is univocal: the global fit works well only if the states  $a_0(1450)$  and  $K_0^*(1430)$  are interpreted as quark-antiquark states. On the contrary, the other combinations deliver large and unacceptable values of  $\chi^2$ , see Table I. We thus conclude that the scalar  $I = 1$  and  $I = 1/2$  states lie above 1 GeV and have to be identified with the resonances  $a_0(1450)$  and  $K_0^*(1430)$ . Moreover, the overall phenomenology described by the fit is very good, see Table II. It is then possible to describe properly many different mesonic masses and decays within a unified treatment based on the symmetries of QCD. It should be stressed that the inclusion of the (axial-)vector mesons has a crucial impact on our results and represents the most important new ingredient of our approach. The good agreement with data also shows that the axial-vector mesons can be interpreted, just as their vector chiral partners, as quark-antiquark states.

We have then studied the consequences of our fit. We have primarily concentrated on the scalar-isoscalar sector which was not included in the fit. In the limit  $N_c \rightarrow \infty$  it is possible to make clear predictions for the two states  $f_0^L$  and  $f_0^H$ . Their masses lie above 1 GeV and their decay patterns have led us to identify  $f_0^L$  with (predominantly)  $f_0(1370)$  and  $f_0^H$  with (predominantly)  $f_0(1710)$ . The theoretical decay rates of  $f_0^L$  are in agreement with experiment;  $f_0^H$  decays only into kaons, but turns out to be too wide. At a qualitative level, this large- $N_c$  result clearly shows that also the scalar-isoscalar quark-antiquark states lie above 1 GeV. The overall situation in the scalar-isoscalar sector can be slightly improved when studying the influence of large- $N_c$  suppressed terms. However, in this system the inclusion of an additional glueball state would be necessary for a full quantitative study. Mixing phenomena are known to be large here and affect both the masses and decays. The study of a genuine three-state system is mandatory: starting from  $\sigma_N = \sqrt{1/2}(\bar{u}u + \bar{d}d)$ ,  $G = gg$ , and  $\sigma_S = \bar{s}s$  to describe properly  $f_0(1370)$ ,  $f_0(1500)$ , and  $f_0(1710)$ . Additionally, we have studied other consequences of the fit, such as predictions for the  $\eta\eta$  channel of scalar-isoscalar states, the  $\eta$ - $\eta'$  mixing, the prediction for the axial-vector kaonic state, and for the  $a_0$  branching ratios. Finally, we have also discussed the origin of the mass of the  $\rho$  meson and, at a qualitative level, its possible value at nuclear matter density.

There is, however, one question that remains open. Firstly, interpreting resonances above 1 GeV as  $\bar{q}q$  states leads to questions about the nature of  $f_0(500)$ ,  $a_0(980)$ ,  $f_0(980)$ , and  $K_0^*(800)$ . Their presence is necessary for the correct description of  $\pi\pi$  scattering lengths [see, e.g. Ref. [15]]. There are two possibilities: (i) they can arise as (quasi-)molecular states of  $\pi\pi$  or  $KK$ , respectively, or (ii) as tetraquark states [49]. The question is whether the attraction is large enough so that these states are bound, or whether the attraction is not sufficient so that they are (unstable) resonances in the continuum. This question and deciding whether possibility (i) or (ii) is realized in nature represent interesting starting points for future studies, along the lines of the Bethe-Salpeter approach of Ref. [50], the lattice approach of Ref. [51], or even holographic approaches such as those presented in Ref. [52]. In either case, one may include the low-lying isoscalar states as interpolating fields in our Lagrangian, such as has been done in Ref. [53].

The present model can be also studied in the baryonic sector, see Ref. [16] for the two-flavor case. The very same ideas of chiral symmetry and dilatation invariance can be applied to the baryonic sector as well. The extension of the model to three flavors in the baryonic sector would surely represent an interesting problem: a multitude of data on decays and masses is available to make a precise test of our approach.

Additionally, restoration of chiral symmetry at nonzero temperature and density is one of the fundamental questions of modern hadron physics, see, e.g. Refs. [10, 54], in which the two-flavor version of this model has been studied at nonzero density, or Ref. [55] for a holographic approach to the exploration of nonzero chemical potential. Linear sigma models are appropriate to study chiral symmetry restoration because they contain from the onset not only (pseudo)scalar and (axial-)vector mesons but also their chiral partners; mass degeneration of chiral partners represents a signal for the chiral transition. Therefore, we also plan to apply the model to study chiral symmetry restoration at nonzero temperatures and densities.

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### Appendix A: Tree-level masses

After spontaneous symmetry breaking (see Sec. II B), from the quadratic terms of the Lagrangian the (squared) mass matrices for the different fields are read off as

$$(m_S^2)_{ij} = S_{ip}(m_0^2\delta_{pq} + 4F_{pqlm}\phi_l\phi_m)S_{qj}, \quad (\text{A1})$$

$$(m_P^2)_{ij} = S_{ip}(m_0^2\delta_{pq} - 9c_1G_{plm}\phi_l\phi_mG_{ql'm'}\phi_{l'}\phi_{m'} + 4H_{pq,lm}\phi_l\phi_m)S_{qj}, \quad (\text{A2})$$

$$(m_V^2)_{ij} = S_{ip}(m_1^2\delta_{pq} + J_{pqlm}\phi_l\phi_m)S_{qj}, \quad (\text{A3})$$

$$(m_A^2)_{ij} = S_{ip}(m_1^2\delta_{pq} + J'_{pqlm}\phi_l\phi_m)S_{qj}, \quad (\text{A4})$$

where  $S_{qj}$  is the  $(9 \times 9)$  transformation matrix to the non strange – strange base,

$$S = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & 0 & \dots & 1 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & -\sqrt{2} \end{pmatrix}, \quad (\text{A5})$$

and the  $F, H, G, J, J'$  coefficient tensors are

$$F_{ijkl} = \frac{\lambda_1}{4}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\lambda_2}{8}(d_{ijm}d_{klm} + d_{ikm}d_{jlm} + d_{ilm}d_{jkm}), \quad (\text{A6})$$

$$H_{ij,kl} = \frac{\lambda_1}{4}\delta_{ij}\delta_{kl} + \frac{\lambda_2}{8}(d_{ijm}d_{klm} + f_{ikm}f_{jlm} + f_{ilm}f_{jkm}), \quad (\text{A7})$$

$$G_{ijk} = \frac{1}{6} \left[ d_{ijk} + \frac{9}{2}d_{000}\delta_{i0}\delta_{j0}\delta_{k0} - \frac{3}{2}(\delta_{i0}d_{jk0} + \delta_{j0}d_{ik0} + \delta_{k0}d_{ij0}) \right], \quad (\text{A8})$$

$$J_{ijkl} = g_1^2 f_{ikm}f_{jlm} + \frac{h_1}{2}\delta_{ij}\delta_{kl} + \frac{h_2}{2}d_{ijm}d_{klm} + \frac{h_3}{4}(d_{ikm}d_{jlm} + d_{ilm}d_{jkm} - f_{ikm}f_{jlm} - f_{ilm}f_{jkm}), \quad (\text{A9})$$

$$J'_{ijkl} = g_1^2 d_{ikm}d_{jlm} + \frac{h_1}{2}\delta_{ij}\delta_{kl} + \frac{h_2}{2}d_{ijm}d_{klm} - \frac{h_3}{4}(d_{ikm}d_{jlm} + d_{ilm}d_{jkm} - f_{ikm}f_{jlm} - f_{ilm}f_{jkm}). \quad (\text{A10})$$

As can be seen from Eqs. (17)–(19), and Eqs. (22)–(24), there is a mixing in the pseudoscalar and scalar  $N-S$  sectors, which can be resolved by the following two-dimensional orthogonal transformations

$$O_{\eta/\sigma} = \begin{pmatrix} \cos \theta_{\eta/\sigma} & \sin \theta_{\eta/\sigma} \\ -\sin \theta_{\eta/\sigma} & \cos \theta_{\eta/\sigma} \end{pmatrix}, \quad (\text{A11})$$

where  $\theta_{\eta/\sigma}$  are the pseudoscalar and scalar mixing angles. The (squared) mass eigenvalues can be written with the help of the mixing angles as

$$m_{\eta_1/\sigma_1}^2 = m_{\eta_N/\sigma_N}^2 \cos^2 \theta_{\eta/\sigma} + m_{\eta_{NS}/\sigma_{NS}}^2 \sin 2\theta_{\eta/\sigma} + m_{\eta_S/\sigma_S}^2 \sin^2 \theta_{\eta/\sigma}, \quad (\text{A12})$$

$$m_{\eta_2/\sigma_2}^2 = m_{\eta_N/\sigma_N}^2 \sin^2 \theta_{\eta/\sigma} - m_{\eta_{NS}/\sigma_{NS}}^2 \sin 2\theta_{\eta/\sigma} + m_{\eta_S/\sigma_S}^2 \cos^2 \theta_{\eta/\sigma},$$

where, if we require that  $m_{\eta_2/\sigma_2}^2 > m_{\eta_1/\sigma_1}^2$ , the mixing angles are given by

$$\sin \theta_{\eta/\sigma} = -\text{sign}(m_{\eta_{NS}/\sigma_{NS}}^2) \frac{1}{\sqrt{2}} \sqrt{1 - \frac{m_{\eta_S/\sigma_S}^2 - m_{\eta_N/\sigma_N}^2}{\sqrt{(m_{\eta_N/\sigma_N}^2 - m_{\eta_S/\sigma_S}^2)^2 + 4m_{\eta_{NS}/\sigma_{NS}}^4}}}, \quad (\text{A13})$$

$$\cos \theta_{\eta/\sigma} = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m_{\eta_S/\sigma_S}^2 - m_{\eta_N/\sigma_N}^2}{\sqrt{(m_{\eta_N/\sigma_N}^2 - m_{\eta_S/\sigma_S}^2)^2 + 4m_{\eta_{NS}/\sigma_{NS}}^4}}} . \quad (\text{A14})$$

It can be seen that, if  $m_{\eta_{NS}/\sigma_{NS}}^2 > 0$ , then  $-\pi/2 < \theta_{\eta/\sigma} < 0$  and, if  $m_{\eta_{NS}/\sigma_{NS}}^2 < 0$ , then  $0 < \theta_{\eta/\sigma} < \pi/2$ . Substituting these expression into Eq. (A12) it is found that

$$m_{\eta_1/\sigma_1}^2 = \frac{1}{2} \left[ m_{\eta_N/\sigma_N}^2 + m_{\eta_S/\sigma_S}^2 - \sqrt{(m_{\eta_N/\sigma_N}^2 - m_{\eta_S/\sigma_S}^2)^2 + 4m_{\eta_{NS}/\sigma_{NS}}^4} \right], \quad (\text{A15})$$

$$m_{\eta_2/\sigma_2}^2 = \frac{1}{2} \left[ m_{\eta_N/\sigma_N}^2 + m_{\eta_S/\sigma_S}^2 + \sqrt{(m_{\eta_N/\sigma_N}^2 - m_{\eta_S/\sigma_S}^2)^2 + 4m_{\eta_{NS}/\sigma_{NS}}^4} \right], \quad (\text{A16})$$

from which it is obvious that the condition  $m_{\eta_2/\sigma_2}^2 > m_{\eta_1/\sigma_1}^2$  is fulfilled. In this way we know that  $m_{\eta_2}$  must be identified as  $m_{\eta'}$ , and similarly for  $\sigma$ .

## Appendix B: Decay widths

In this section we show the explicit form of some of the most relevant decay widths calculated from our model at tree-level using Eqs. (38). The formulas below are organized according to the type of the decaying particle.

### 1. Scalar-meson decay widths

At first we are considering the decays of the scalar isotriplet  $a_0$ , scalar kaon  $K_0^*$ , and scalar isosinglets  $f_0^{H/L}$ . The  $a_0$  state has three relevant decay channels, namely into  $\eta\pi$ ,  $\eta'\pi$ , and  $KK$ , the first two of which are strongly connected due to the mixing between  $\eta$  and  $\eta'$ . The  $a_0 \rightarrow \eta\pi$  and  $a_0 \rightarrow \eta'\pi$  decay widths<sup>4</sup>, considering that in both cases  $\mathcal{I} = 1$ , read

$$\Gamma_{a_0 \rightarrow \eta\pi} = \frac{1}{8m_{a_0}\pi} \left[ \frac{(m_{a_0}^2 - m_\eta^2 - m_\pi^2)^2 - 4m_\eta^2 m_\pi^2}{4m_{a_0}^4} \right]^{1/2} |\mathcal{M}_{a_0 \rightarrow \eta\pi}|^2, \quad (\text{B1})$$

$$\Gamma_{a_0 \rightarrow \eta'\pi} = \frac{1}{8m_{a_0}\pi} \left[ \frac{(m_{a_0}^2 - m_{\eta'}^2 - m_\pi^2)^2 - 4m_{\eta'}^2 m_\pi^2}{4m_{a_0}^4} \right]^{1/2} |\mathcal{M}_{a_0 \rightarrow \eta'\pi}|^2, \quad (\text{B2})$$

with the following transition matrix elements,

$$\mathcal{M}_{a_0 \rightarrow \eta\pi} = \cos \theta_\pi \mathcal{M}_{a_0 \rightarrow \eta_N \pi}(m_\eta) + \sin \theta_\pi \mathcal{M}_{a_0 \rightarrow \eta_S \pi}(m_\eta), \quad (\text{B3})$$

$$\mathcal{M}_{a_0 \rightarrow \eta'\pi} = \cos \theta_\pi \mathcal{M}_{a_0 \rightarrow \eta_S \pi}(m_{\eta'}) - \sin \theta_\pi \mathcal{M}_{a_0 \rightarrow \eta_N \pi}(m_{\eta'}), \quad (\text{B4})$$

where

$$\mathcal{M}_{a_0 \rightarrow \eta_N \pi}(m) = A_{a_0 \eta_N \pi} - B_{a_0 \eta_N \pi} \frac{m_{a_0}^2 - m^2 - m_\pi^2}{2} + C_{a_0 \eta_N \pi} m_{a_0}^2, \quad (\text{B5})$$

$$\mathcal{M}_{a_0 \rightarrow \eta_S \pi}(m) = A_{a_0 \eta_S \pi}, \quad (\text{B6})$$

and

$$A_{a_0 \eta_N \pi} = Z_\pi^2 (c_1 \phi_S^2 - \lambda_2) \phi_N, \quad (\text{B7})$$

$$B_{a_0 \eta_N \pi} = -2 \frac{g_1^2 \phi_N}{m_{a_1}^2} \left[ 1 - \frac{1}{2} \frac{Z_\pi^2 \phi_N^2}{m_{a_1}^2} (h_2 - h_3) \right], \quad (\text{B8})$$

$$C_{a_0 \eta_N \pi} = g_1 Z_\pi^2 w_{a_1}, \quad (\text{B9})$$

---

<sup>4</sup> Obviously, the neutral and the charged  $a_0$ 's have the same formulas for the decay widths.



$$A_{a_0\eta_S\pi} = \frac{1}{2}c_1 Z_\pi Z_{\eta_S} \phi_N^2 \phi_S. \quad (\text{B10})$$

The  $a_0 \rightarrow KK$  decay width includes two subchannels,  $K^0 \bar{K}^0$  and  $K^- K^+$ , which results in the isospin factor  $\mathcal{I} = 2$ , and accordingly the decay width is found to be

$$\Gamma_{a_0 \rightarrow KK} = \frac{1}{8m_{a_0}\pi} \sqrt{1 - \left(\frac{2m_K}{m_{a_0}}\right)^2} \left| A_{a_0 KK} - \frac{1}{2} B_{a_0 KK} (m_{a_0}^2 - 2m_K^2) + C_{a_0 KK} m_{a_0}^2 \right|^2, \quad (\text{B11})$$

where

$$A_{a_0 KK} = \lambda_2 Z_K^2 \left( \phi_N - \frac{\phi_S}{\sqrt{2}} \right), \quad (\text{B12})$$

$$B_{a_0 KK} = Z_K^2 w_{K_1} \left\{ g_1 - \frac{1}{2} w_{K_1} \left( (g_1^2 + h_2) \phi_N + \sqrt{2} (g_1^2 - h_3) \phi_S \right) \right\}, \quad (\text{B13})$$

$$C_{a_0 KK} = -\frac{g_1}{2} Z_K^2 w_{K_1}. \quad (\text{B14})$$

Now turning to the scalar kaon there is only one relevant decay channel,  $K_0^*(\text{or } K_0) \rightarrow K\pi$ , for which the decay width reads

$$\Gamma_{K_0 \rightarrow K\pi} = \frac{3}{8\pi m_{K_0}} \left[ \frac{(m_{K_0}^2 - m_\pi^2 - m_K^2)^2 - 4m_\pi^2 m_K^2}{4m_{K_0}^4} \right]^{1/2} \left[ A_{K_0 K\pi} + (C_{K_0 K\pi} + D_{K_0 K\pi} - B_{K_0 K\pi}) \right. \\ \left. \times \frac{m_{K_0}^2 - m_K^2 - m_\pi^2}{2} + C_{K_0 K\pi} m_K^2 + D_{K_0 K\pi} m_\pi^2 \right], \quad (\text{B15})$$

where

$$A_{K_0 K\pi} = \frac{Z_\pi Z_K Z_{K_0}}{\sqrt{2}} \lambda_2 \phi_S, \quad (\text{B16})$$

$$B_{K_0 K\pi} = \frac{Z_\pi Z_K Z_{K_0}}{4} w_{a_1} w_{K_1} \left[ 2g_1 \frac{w_{a_1} + w_{K_1}}{w_{a_1} w_{K_1}} + (2h_3 - h_2 - 3g_1^2) \phi_N - \sqrt{2} (g_1^2 + h_2) \phi_S \right], \quad (\text{B17})$$

$$C_{K_0 K\pi} = \frac{Z_\pi Z_K Z_{K_0}}{2} [-g_1 (i w_{K^*} + w_{K_1}) + \sqrt{2} i w_{K^*} w_{K_1} (g_1^2 - h_3) \phi_S], \quad (\text{B18})$$

$$D_{K_0 K\pi} = \frac{Z_\pi Z_K Z_{K_0}}{4} [2g_1 (i w_{K^*} - w_{a_1}) + i w_{K^*} w_{a_1} ((2h_3 - h_2 - 3g_1^2) \phi_N + \sqrt{2} (g_1^2 + h_2) \phi_S)]. \quad (\text{B19})$$

Finally, for the  $f_0^{L/H}$  particles there are two relevant decay channels: the two-pion and the two-kaon channels. It is important to note that due to the particle mixing between  $f_0^L$  and  $f_0^H$ , the matrix elements are linear combinations that depend on the scalar mixing angle  $\theta_\sigma$  (see Appendix A), as can be seen explicitly below. The decay widths in the  $\pi\pi$  channel are

$$\Gamma_{f_0^L \rightarrow \pi\pi} = \frac{3}{32\pi m_{f_0^L}} \sqrt{1 - \left(\frac{2m_\pi}{m_{f_0^L}}\right)^2} \left| \mathcal{M}_{f_0^L \rightarrow \pi\pi} \right|^2, \quad (\text{B20})$$

$$\Gamma_{f_0^H \rightarrow \pi\pi} = \frac{3}{32\pi m_{f_0^H}} \sqrt{1 - \left(\frac{2m_\pi}{m_{f_0^H}}\right)^2} \left| \mathcal{M}_{f_0^H \rightarrow \pi\pi} \right|^2, \quad (\text{B21})$$

where an isospin factor of 3/2 was considered<sup>5</sup> and the matrix elements are

$$\mathcal{M}_{f_0^L \rightarrow \pi\pi} = -\sin \theta_\sigma \mathcal{M}_{f_0\pi}^H(m_{f_0^L}) + \cos \theta_\sigma \mathcal{M}_{f_0\pi}^L(m_{f_0^L}), \quad (\text{B22})$$

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<sup>5</sup> There are two subchannels, namely the  $\pi^+\pi^-$  and the  $\pi^0\pi^0$ , which would mean  $\mathcal{I} = 2$ . However, since the two  $\pi^0$  are indistinguishable, the solid-angle integral (there is no angular dependence at tree-level) ends up as  $2\pi$  instead of  $4\pi$ , which means a factor of 1/2 in case of  $\pi^0\pi^0$  compared to  $\pi^+\pi^-$ , thus  $\mathcal{I} = 1/2 + 1$ .

$$\mathcal{M}_{f_0^H \rightarrow \pi\pi} = \cos\theta_\sigma \mathcal{M}_{f_0\pi}^H(m_{f_0^H}) + \sin\theta_\sigma \mathcal{M}_{f_0\pi}^L(m_{f_0^H}), \quad (\text{B23})$$

$$\mathcal{M}_{f_0\pi}^L(m) = 2Z_\pi^2 \phi_N \left\{ \frac{g_1^2}{2} \frac{m^2}{m_{a_1}^2} \left[ 1 + \left( 1 - \frac{2m_\pi^2}{m^2} \right) \frac{m_1^2 + h_1 \phi_S^2/2 + 2\delta_N}{m_{a_1}^2} \right] - \left( \lambda_1 + \frac{\lambda_2}{2} \right) \right\}, \quad (\text{B24})$$

$$\mathcal{M}_{f_0\pi}^H(m) = 2Z_\pi^2 \phi_S \left\{ -\frac{g_1^2}{4} \frac{m^2}{m_{a_1}^2} \left( 1 - \frac{2m_\pi^2}{m^2} \right) \frac{h_1 \phi_N^2}{m_{a_1}^2} - \lambda_1 \right\}. \quad (\text{B25})$$

In the  $KK$  channel, where  $\mathcal{I} = 2$ , the decay widths read

$$\Gamma_{f_0^H \rightarrow KK} = \frac{1}{8\pi m_{f_0^H}} \sqrt{1 - \left( \frac{2m_K}{m_{f_0^H}} \right)^2} \left| \mathcal{M}_{f_0^H \rightarrow KK} \right|^2, \quad (\text{B26})$$

$$\Gamma_{f_0^L \rightarrow KK} = \frac{1}{8\pi m_{f_0^L}} \sqrt{1 - \left( \frac{2m_K}{m_{f_0^L}} \right)^2} \left| \mathcal{M}_{f_0^L \rightarrow KK} \right|^2, \quad (\text{B27})$$

where the matrix elements, using the notations  $H_N \equiv (g_1^2 + 2h_1 + h_2)/4$  and  $H_S \equiv (g_1^2 + h_1 + h_2)/2$ , are

$$\mathcal{M}_{f_0^L \rightarrow KK} = -\sin\theta_\sigma \mathcal{M}_{f_0K}^H(m_{f_0^L}) + \cos\theta_\sigma \mathcal{M}_{f_0K}^L(m_{f_0^L}), \quad (\text{B28})$$

$$\mathcal{M}_{f_0^H \rightarrow KK} = \cos\theta_\sigma \mathcal{M}_{f_0K}^H(m_{f_0^H}) + \sin\theta_\sigma \mathcal{M}_{f_0K}^L(m_{f_0^H}), \quad (\text{B29})$$

$$\mathcal{M}_{f_0K}^L(m) = -Z_K^2 \left[ (2\lambda_1 + \lambda_2)\phi_N - \frac{\lambda_2}{\sqrt{2}}\phi_S + g_1 w_{K_1}(m_K^2 - m^2) + w_{K_1}^2 \left( 2H_N \phi_N - \frac{h_3 - g_1^2}{\sqrt{2}} \phi_S \right) \frac{m^2 - 2m_K^2}{2} \right], \quad (\text{B30})$$

$$\mathcal{M}_{f_0K}^H(m) = -Z_K^2 \left[ 2(\lambda_1 + \lambda_2)\phi_S - \frac{\lambda_2}{\sqrt{2}}\phi_N + \sqrt{2}g_1 w_{K_1}(m_K^2 - m^2) + w_{K_1}^2 \left( 2H_S \phi_S - \frac{h_3 - g_1^2}{\sqrt{2}} \phi_N \right) \frac{m^2 - 2m_K^2}{2} \right]. \quad (\text{B31})$$

## 2. Vector-meson decay widths

In the case of the vector mesons we are considering the decays of the  $\rho$  meson, the  $K^*$  vector kaon, and the  $\phi$  meson. All these particles have only one relevant decay channel. The first is the  $\rho \rightarrow \pi\pi$  decay which has the following quite simple decay width formula

$$\Gamma_{\rho \rightarrow \pi\pi} = \frac{m_\rho^5}{48\pi m_{a_1}^4} \left[ 1 - \left( \frac{2m_\pi}{m_\rho} \right)^2 \right]^{3/2} \left[ g_1 Z_\pi^2 - \frac{g_2}{2} (Z_\pi^2 - 1) \right]^2. \quad (\text{B32})$$

The next one is the  $K^* \rightarrow K\pi$  decay, in case of which the decay width reads

$$\Gamma_{K^* \rightarrow K\pi} = \frac{m_{K^*}}{8\pi} \left[ \frac{(m_{K^*}^2 - m_\pi^2 - m_K^2)^2 - 4m_\pi^2 m_K^2}{4m_{K^*}^4} \right]^{3/2} (A_{K^*K\pi} - B_{K^*K\pi} + C_{K^*K\pi})^2, \quad (\text{B33})$$

where the constants are defined as

$$\begin{aligned} A_{K^*K\pi} &= \frac{1}{2} Z_\pi Z_K \left[ -g_1 + \sqrt{2} w_{K_1} (g_1^2 - h_3) \phi_S \right], \\ B_{K^*K\pi} &= \frac{1}{4} Z_\pi Z_K \left[ 2g_1 + w_{a_1} (-3g_1^2 - h_2 + 2h_3) \phi_N + \sqrt{2} w_{a_1} (g_1^2 + h_2) \phi_S \right], \\ C_{K^*K\pi} &= \frac{1}{2} Z_\pi Z_K w_{a_1} w_{K_1} g_2 m_{K^*}^2. \end{aligned} \quad (\text{B34})$$

Finally, the  $\phi \rightarrow KK$  decay width reads

$$\Gamma_{\phi \rightarrow KK} = \frac{m_\phi^5}{192\pi m_{K_1}^4} \left[ 1 - \left( \frac{2m_K}{m_\phi} \right)^2 \right]^{3/2} \left[ 2g_1 Z_K^2 \left( 1 + \frac{\delta_N - \delta_S}{m_\phi^2} \right) - g_2 (Z_K^2 - 1) \right]^2. \quad (\text{B35})$$

### 3. Axial-vector-meson decay widths

Turning to axial-vector mesons, the considered decays are the  $a_1$  decay with two relevant channels and the  $f_{1S}$  decay with one relevant channel. Since in case of the  $a_1 \rightarrow \rho\pi$  and  $f_{1S} \rightarrow KK^*$  decays, the decaying as well as one of the resulting particle are (axial-)vector mesons, the matrix elements have a more complicated form than in the previous cases, as can be seen below. The first considered decay width is the one of the  $a_1 \rightarrow \pi\gamma$  process, which takes the following simple form

$$\Gamma_{a_1 \rightarrow \pi\gamma} = \frac{e^2 g_1^2 \phi_N^2}{96\pi m_{a_1}} Z_\pi^2 \left[ 1 - \left( \frac{m_\pi}{m_{a_1}} \right)^2 \right]^3. \quad (\text{B36})$$

In case of the  $a_1 \rightarrow \rho\pi$  decay width one has to consider two channels ( $\rho^+\pi^-$ ,  $\rho^-\pi^+$ ), thus  $\mathcal{I} = 2$  and the decay width is found to be

$$\Gamma_{a_1 \rightarrow \rho\pi} = \frac{1}{12m_{a_1}\pi} \left[ \frac{(m_{a_1}^2 - m_\rho^2 - m_\pi^2)^2 - 4m_\rho^2 m_\pi^2}{4m_{a_1}^4} \right]^{1/2} \left[ |V_{\mu\nu}|^2 - \frac{|V_{\mu\nu} k_\rho^\nu|^2}{m_\rho^2} - \frac{|V_{\mu\nu} k_{a_1}^\mu|^2}{m_{a_1}^2} + \frac{|V_{\mu\nu} k_{a_1}^\mu k_\rho^\nu|^2}{m_\rho^2 m_{a_1}^2} \right], \quad (\text{B37})$$

where  $V_{\mu\nu}$  is the vertex following from the relevant part of the Lagrangian,

$$V_{\mu\nu} = iZ_\pi \phi_N \left\{ (g_1^2 - h_3) g_{\mu\nu} + \frac{g_1 g_2}{m_{a_1}^2} [k_{\pi\mu} k_{a_1\nu} + k_{\rho\mu} k_{\pi\nu} - k_\pi \cdot (k_\rho + k_{a_1}) g_{\mu\nu}] \right\} \quad (\text{B38})$$

and  $k_{a_1}^\mu = (m_{a_1}, \mathbf{0})$ ,  $k_\rho^\mu = (E_\rho, \mathbf{k})$  and  $k_\pi^\mu = (E_\pi, -\mathbf{k})$  are the four-momenta of  $a_1$ ,  $\rho$ , and  $\pi$  in the rest frame of  $a_1$ , respectively. Using the following kinematic relations

$$\begin{aligned} k_\pi \cdot k_\rho &= \frac{m_{a_1}^2 - m_\rho^2 - m_\pi^2}{2}, \\ k_{a_1} \cdot k_\pi &= m_{a_1} E_\pi = \frac{m_{a_1}^2 + m_\pi^2 - m_\rho^2}{2}, \\ k_{a_1} \cdot k_\rho &= m_{a_1} E_\rho = \frac{m_{a_1}^2 + m_\rho^2 - m_\pi^2}{2}, \\ \mathbf{k}^2 &= \frac{(m_{a_1}^2 - m_\pi^2 - m_\rho^2)^2 - 4m_\pi^2 m_\rho^2}{4m_{a_1}^2}, \end{aligned} \quad (\text{B39})$$

the terms in Eq. (B37) are given by

$$\begin{aligned} |V_{\mu\nu}|^2 &= Z_\pi^2 \phi_N^2 \left\{ 4(g_1^2 - h_3)^2 + \frac{g_1^2 g_2^2}{m_{a_1}^4} \left[ \frac{5}{2}(m_{a_1}^2 - m_\rho^2)^2 + \frac{1}{2}m_\pi^2(2m_{a_1}^2 + 2m_\rho^2 - m_\pi^2) \right] - 6 \frac{g_1 g_2 (g_1^2 - h_3)}{m_{a_1}^2} (m_{a_1}^2 - m_\rho^2) \right\}, \\ \frac{|V_{\mu\nu} k_\rho^\nu|^2}{m_\rho^2} &= Z_\pi^2 \phi_N^2 \left\{ (g_1^2 - h_3)^2 - \frac{g_1^2 g_2^2}{m_\rho^2} (E_\rho^2 - m_\rho^2) + 2 \frac{g_1 g_2 (g_1^2 - h_3)}{m_\rho^2} (E_\pi^2 - m_\pi^2) \right\}, \\ \frac{|V_{\mu\nu} k_{a_1}^\nu|^2}{m_{a_1}^2} &= Z_\pi^2 \phi_N^2 \left\{ (g_1^2 - h_3)^2 - \frac{g_1^2 g_2^2}{m_{a_1}^4} m_\rho^2 \mathbf{k}^2 - 2 \frac{g_1 g_2 (g_1^2 - h_3)}{m_{a_1}^2} (E_\rho^2 - m_\rho^2) \right\}, \\ \frac{|V_{\mu\nu} k_{a_1}^\mu k_\rho^\nu|^2}{m_\rho^2 m_{a_1}^2} &= Z_\pi^2 \phi_N^2 \frac{(g_1^2 - h_3)^2}{m_\rho^2} E_\rho^2. \end{aligned} \quad (\text{B40})$$

Analogously to the previous case, the width of the  $f_{1S} \rightarrow KK^*$  decay which includes four sub-channels ( $K^0 \bar{K}^{*0}$ ,  $\bar{K}^0 K^{*0}$ ,  $K^- K^{*+}$ ,  $K^+ K^{*-}$ ), resulting in  $\mathcal{I} = 4$ , becomes

$$\Gamma_{f_{1S} \rightarrow KK^*} = \frac{1}{6m_{f_{1S}}\pi} \left[ \frac{(m_{f_{1S}}^2 - m_{K^*}^2 - m_K^2)^2 - 4m_{K^*}^2 m_K^2}{4m_{f_{1S}}^4} \right]^{1/2} \left[ |\tilde{V}_{\mu\nu}|^2 - \frac{|\tilde{V}_{\mu\nu} k_{K^*}^\nu|^2}{m_{K^*}^2} - \frac{|\tilde{V}_{\mu\nu} k_{f_{1S}}^\mu|^2}{m_{f_{1S}}^2} + \frac{|\tilde{V}_{\mu\nu} k_{f_{1S}}^\mu k_{K^*}^\nu|^2}{m_{K^*}^2 m_{f_{1S}}^2} \right], \quad (\text{B41})$$

where  $\tilde{V}_{\mu\nu}$  has the same Lorentz structure as Eq. (B38) and the only difference is in the constants in front of the different terms. More explicitly,

$$\tilde{V}_{\mu\nu} = iZ_K \{ A_{f_{1S}KK} g_{\mu\nu} + B_{f_{1S}KK} [k_{K\mu} k_{f_{1S}\nu} + k_{K^*\mu} k_{K\nu} - k_K \cdot (k_{K^*} + k_{f_{1S}}) g_{\mu\nu}] \}, \quad (\text{B42})$$

$$A_{fKK} = \frac{1}{4} \left[ g_1^2 (\sqrt{2}\phi_N - 6\phi_S) + \sqrt{2}h_2(\phi_N - \sqrt{2}\phi_S) + 4h_3\phi_S \right], \quad (\text{B43})$$

$$B_{fKK} = -\frac{1}{\sqrt{2}}g_2w_{K_1}. \quad (\text{B44})$$

The kinematic relations are the same as in Eq. (B39) with the following substitutions:  $a_1 \rightarrow f_{1S}, \pi \rightarrow K, \rho \rightarrow K^*$ , while the expressions analogous to Eq. (B40) are,

$$\begin{aligned} |\tilde{V}_{\mu\nu}|^2 &= Z_K^2 \left\{ 4A_{fKK}^2 + B_{fKK}^2 \left[ \frac{5}{2}(m_{f_{1S}}^2 - m_{K^*}^2)^2 + \frac{1}{2}m_K^2(2m_{f_{1S}}^2 + 2m_{K^*}^2 - m_K^2) \right] - 6A_{fKK}B_{fKK}(m_{f_{1S}}^2 - m_{K^*}^2) \right\}, \\ \frac{|\tilde{V}_{\mu\nu}k_{K^*}^\nu|^2}{m_{K^*}^2} &= Z_K^2 \{ A_{fKK}^2 - B_{fKK}^2(E_{K^*}^2 - m_{K^*}^2) + 2A_{fKK}B_{fKK}(E_K^2 - m_K^2) \}, \\ \frac{|\tilde{V}_{\mu\nu}k_{f_{1S}}^\nu|^2}{m_{f_{1S}}^2} &= Z_K^2 \{ A_{fKK}^2 - B_{fKK}^2m_{K^*}^2\mathbf{k}^2 - 2A_{fKK}B_{fKK}(E_{K^*}^2 - m_{K^*}^2) \}, \\ \frac{|\tilde{V}_{\mu\nu}k_{f_{1S}}^\mu k_{K^*}^\nu|^2}{m_{K^*}^2 m_{f_{1S}}^2} &= Z_K^2 \frac{A_{fKK}^2}{m_{K^*}^2} E_{K^*}^2. \end{aligned} \quad (\text{B45})$$

Analogously, the width for a generic decay of the form  $K_1 \rightarrow VP$ , where  $V$  denotes a vector and  $P$  a pseudoscalar state, reads

$$\Gamma_{K_1 \rightarrow VP} = \mathcal{I} \frac{k(m_{K_1}, m_V, m_P)}{8\pi m_{K_1}^2} \frac{1}{3} \left[ \left| \tilde{V}_{K_1 VP}^{\mu\nu} \right|^2 - \frac{\left| \tilde{V}_{K_1 VP}^{\mu\nu} p_{K_1\mu} \right|^2}{m_{K_1}^2} - \frac{\left| \tilde{V}_{K_1 VP}^{\mu\nu} p_{V\nu} \right|^2}{m_V^2} + \frac{\left| \tilde{V}_{K_1 VP}^{\mu\nu} p_{K_1\mu} p_{V\nu} \right|^2}{m_V^2 m_{K_1}^2} \right], \quad (\text{B46})$$

where  $\mathcal{I} = 3$  for the  $K_1 \rightarrow \rho K$  and  $K_1 \rightarrow K^* \pi$  decays and  $\mathcal{I} = 1$  for the  $K_1 \rightarrow \omega_N K$  decay,

$$k(m_a, m_b, m_c) = \frac{1}{2m_a} \sqrt{m_a^4 - 2m_a^2(m_b^2 + m_c^2) + (m_b^2 - m_c^2)^2} \theta(m_a - m_b - m_c), \quad (\text{B47})$$

(the theta function ensures that the decay width vanishes below threshold) and

$$\tilde{V}_{K_1 VP}^{\mu\nu} = i \{ A_{K_1} g^{\mu\nu} + B_{K_1} [p_V^\mu p_P^\nu + p_P^\mu p_{K_1}^\nu - (p_P \cdot p_V) g^{\mu\nu} - (p_{K_1} \cdot p_P) g^{\mu\nu}] \}. \quad (\text{B48})$$

The values of  $A_{K_1}$ ,  $B_{K_1}$ , and  $C_{K_1}$  depend on the process considered:  $A_{K_1} = \{A_{K_1 K^* \pi}, A_{K_1 \rho K}, A_{K_1 \omega_N K}\}$  and  $B_{K_1} = \{B_{K_1 K^* \pi}, B_{K_1 \rho K}, B_{K_1 \omega_N K}\}$ . The coefficients read

$$A_{K_1 K^* \pi} = \frac{i}{\sqrt{2}} Z_\pi (h_3 - g_1^2) \phi_S, \quad (\text{B49})$$

$$B_{K_1 K^* \pi} = -\frac{i}{2} Z_\pi g_2 w_{a_1} \quad (\text{B50})$$

$$A_{K_1 \rho K} = \frac{i}{4} Z_K \left[ g_1^2 (\phi_N + \sqrt{2}\phi_S) - h_2 (\phi_N - \sqrt{2}\phi_S) - 2h_3 \phi_N \right], \quad (\text{B51})$$

$$B_{K_1 \rho K} = \frac{i}{2} Z_K g_2 w_{K_1}, \quad (\text{B52})$$

$$A_{K_1 \omega_N K} = -\frac{i}{4} Z_K [g_1^2 (\phi_N + \sqrt{2}\phi_S) - h_2 (\phi_N - \sqrt{2}\phi_S) - 2h_3 \phi_N], \quad (\text{B53})$$

$$B_{K_1 \omega_N K} = -\frac{i}{2} Z_K g_2 w_{K_1}. \quad (\text{B54})$$

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